

Solutions

Chapter 1

Concept Application Exercises

Exercise 1.1

1. Clearly, $x = 2$ satisfies the given equation.

For $x < 2$, equation becomes

$$x^3 3^{2-x} + 3^{x+1} = x^3 \cdot 3^{x-2} + 3^{5-x}$$

$$\Rightarrow 3^{2-x} (x^3 - 3^3) = 3^{x-2} (x^3 - 3^3)$$

$$\Rightarrow (x^3 - 3^3) (3^{2-x} - 3^{x-2}) = 0$$

$$\Rightarrow x = 3 \text{ or } x = 2$$

$$\therefore x = 2 \text{ is the only solution}$$

$$2. \left(\frac{1}{2}\right)^{x^6-2x^4} < 2^{x^2}$$

$$\text{or } \left(\frac{1}{2}\right)^{x^6-2x^4} < \left(\frac{1}{2}\right)^{-x^2}$$

$$\text{or } x^6 - 2x^4 > -x^2$$

$$\text{or } x^6 - 2x^4 + x^2 > 0$$

$$\text{or } (x^3 - x)^2 > 0$$

$$\Rightarrow x^3 - x \neq 0$$

$$\therefore x \neq 0, -1, 1$$

$$\therefore x \in \mathbb{R} - \{0, \pm 1\}$$

$$3. y^x = x^y; x = 2y$$

$$\Rightarrow y^x = (2y)^{x/2}$$

$$\Rightarrow y^{x/2} = 2^{x/2}$$

$$\Rightarrow y = 2, \text{ hence } x = 4$$

$$4. 2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$$

$$\Rightarrow 2^x (4 - 8 - 16) > 5^x (5 - 25)$$

$$\Rightarrow (2/5)^x < 1$$

$$\Rightarrow (2/5)^x < (2/5)^0$$

$$\Rightarrow x \in (0, \infty)$$

$$5. \left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$$

$$\Rightarrow \left(\frac{3}{4}\right)^{6x+10-x^2} < \left(\frac{3}{4}\right)^3$$

$$\Rightarrow 6x + 10 - x^2 > 3$$

$$\Rightarrow x^2 - 6x - 7 < 0$$

$$\Rightarrow (x+1)(x-7) < 0$$

$$\Rightarrow -1 < x < 7$$

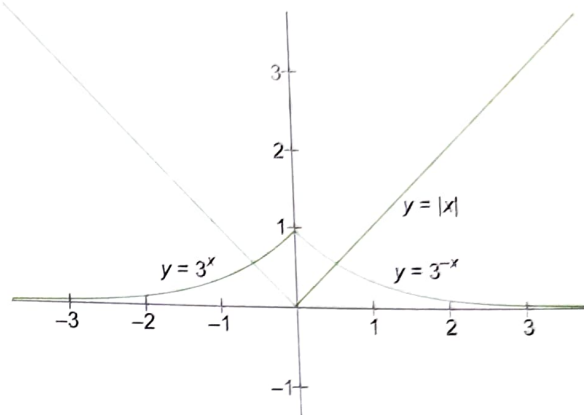
$$6. \text{ We have } |x|3^{|x|} = 1$$

$$\text{or } |x| = 3^{-|x|}$$

$$\text{Now, } 3^{-|x|} = \begin{cases} 3^{-x}, & x \geq 0 \\ 3^x, & x < 0 \end{cases}$$

To find the number of roots of the above equation, we need to find the number of points of intersection of $y = |x|$ and $y = 3^{-|x|}$

The graphs of these functions are as shown in the following figure:



From the graph number of solutions is 2.

Exercise 1.2

$$1. 3^{2\log_9 3} = 3^{2\log_9 9^{1/2}}$$

$$= 3^{2\left(\frac{1}{2}\right)} = 3$$

$$2. \sqrt{(\log_{0.5} 4)^2} = |\log_{0.5} 4|$$

$$= |\log_{0.5} (0.5)^{-2}|$$

$$= |-2|$$

$$= 2$$

$$3. \log_{\sqrt{8}} b = 3 \frac{1}{3}$$

$$\Rightarrow b = (\sqrt{8})^{\frac{10}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{10}{3}} = 2^5 = 32$$

$$4. \log_5 \log_2 \log_3 \log_2 (2^9)$$

$$= \log_5 \log_2 \log_3 9$$

$$= \log_5 \log_2 \log_3 3^2$$

$$= \log_5 \log_2 2$$

$$= \log_5 1$$

$$= 0$$

$$5. \log_5 x = a$$

$$\therefore x = 5^a$$

$$\log_2 y = a$$

$$\therefore y = 2^a$$

$$100^{2a-1} = (5^2 2^2)^{2a} \times 100^{-1}$$

$$= (5^a 2^a)^4 \times 100^{-1}$$

$$= \frac{(xy)^4}{100}$$

$$6. \log_{1/3} \sqrt[4]{729 \times \sqrt[3]{9^{-1} \times 27^{-4/3}}}$$

$$= \log_{1/3} \sqrt[4]{729 \times \sqrt[3]{3^{-2} \times 3^{-4}}}$$

$$\begin{aligned}
 &= \log_{1/3} \sqrt[4]{3^6 \times 3^{-2}} \\
 &= \log_{1/3} 3 \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 7. \log_4 \log_3 \log_2 x &= 0 \\
 \Rightarrow \log_3 \log_2 x &= 1 \\
 \Rightarrow \log_2 x &= 3 \\
 \Rightarrow x &= 2^3 = 8
 \end{aligned}$$

$$8. \text{ We have to prove that } \frac{1}{4} < \log_{10} 2 < \frac{1}{3}.$$

$$\text{Consider } \log_{10} 2 > \frac{1}{4}$$

$$\begin{aligned}
 \Rightarrow 2 &> 10^{\frac{1}{4}} \\
 \Rightarrow 2^4 &> 10, \text{ which is true}
 \end{aligned}$$

$$\text{Now consider } \log_{10} 2 < \frac{1}{3}$$

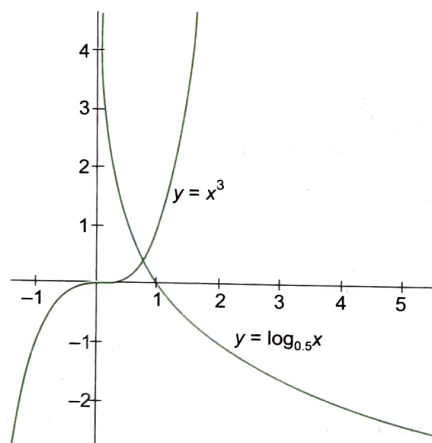
$$\begin{aligned}
 \Rightarrow 2 &< 10^{\frac{1}{3}} \\
 \Rightarrow 2^3 &< 10, \text{ which is true.}
 \end{aligned}$$

$$\text{Thus, } \frac{1}{4} < \log_{10} 2 < \frac{1}{3}$$

$$9. \text{ We have } x^3 - \log_{0.5} x = 0$$

$$\text{or } x^3 = \log_{0.5} x$$

To find the number of roots of the equation, we have to draw the graphs of $y = x^3$ and $y = \log_{0.5} x$.



From the figure, graphs intersect at only one point.
Hence, the equation has only one root.

Exercise 1.3

$$1. (a) 1 + \log_2 5 = \log_2 2 + \log_2 5 = \log_2 (2 \times 5) = \log_2 10$$

$$\begin{aligned}
 (b) 2 - \log_3 7 &= 2\log_3 3 - \log_3 7 \\
 &= \log_3 3^2 - \log_3 7 \\
 &= \log_3 9 - \log_3 7 \\
 &= \log_3 \frac{9}{7}
 \end{aligned}$$

$$\begin{aligned}
 (c) 2\log_{10} x + 3\log_{10} y - 5\log_{10} z \\
 &= \log_{10} x^2 + \log_{10} y^3 - \log_{10} z^5 \\
 &= \log_{10} (x^2 y^3) - \log_{10} z^5 \\
 &= \log_{10} \frac{x^2 y^3}{z^5}
 \end{aligned}$$

$$2. x = \log_{20} 3$$

$$\Rightarrow \frac{1}{x} = \log_3 20$$

We know that $3^2 < 20 < 3^3$

$$\Rightarrow 2 < \log_3 20 < 3$$

$$\Rightarrow 2 < \frac{1}{x} < 3$$

$$\Rightarrow x \in \left(\frac{1}{3}, \frac{1}{2} \right)$$

$$\begin{aligned}
 3. \log_7 \log_7 \sqrt{7\sqrt{7\sqrt{7}}} &= \log_7 \log_7 7^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}} \\
 &= \log_7 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right) \\
 &= \log_7 \left(\frac{7}{8} \right) \\
 &= 1 - \log_7 8 \\
 &= 1 - 3 \log_7 2
 \end{aligned}$$

$$4. \log_{1000} (x^2) = \frac{2}{3} \log_{10} x = \frac{2}{3} y$$

$$\begin{aligned}
 5. \log_{49} 28 &= \log_{(7^2)} (2^2 \times 7) \\
 &= \frac{1}{2} \log_7 (2^2 \times 7) \\
 &= \frac{1}{2} [\log_7 2^2 + \log_7 7] \\
 &= \frac{1}{2} [2\log_7 2 + 1] \\
 &= \frac{1 + 2m}{2}
 \end{aligned}$$

$$6. \log_2 \left(\frac{1}{7^{\log_7 0.125}} \right) = \log_2 \left(\frac{1}{0.125} \right) = \log_2 8 = 3$$

$$\begin{aligned}
 7. \left(\frac{4}{\log_2 (2\sqrt{3})} + \frac{2}{\log_3 (2\sqrt{3})} \right)^2 &= (4\log_{(2\sqrt{3})} 2 + 2\log_{(2\sqrt{3})} 3)^2 \\
 &= (\log_{2\sqrt{3}} 16 + \log_{2\sqrt{3}} 9)^2 \\
 &= (\log_{2\sqrt{3}} 144)^2 \\
 &= (\log_{2\sqrt{3}} (2\sqrt{3})^4)^2 \\
 &= 4^2 = 16
 \end{aligned}$$

$$8. \text{ We have } 2\log(2y - 3x) = \log x + \log y$$

$$\Rightarrow \log(2y - 3x)^2 = \log xy$$

$$\Rightarrow 4y^2 - 12xy + 9x^2 = xy$$

$$\Rightarrow 4y^2 - 13xy + 9x^2 = 0$$

$$\Rightarrow (4y - 9x)(x - y) = 0$$

$$\Rightarrow 4y = 9x$$

$$\Rightarrow \frac{x}{y} = \frac{4}{9}$$

(As for $x = y$, LHS is not defined)

$$9. a^2 + b^2 = 7ab$$

$$\text{or } (a + b)^2 = 9ab$$

$$\text{or } \log(a + b)^2 = \log 9ab$$

$$\text{or } 2\log(a + b) = 2\log 3 + \log a + \log b$$

$$\text{or } 2(\log(a + b) - \log 3) = \log a + \log b$$

$$\text{or } \log \left(\frac{a + b}{3} \right) = \frac{1}{2} (\log a + \log b)$$

10. Eliminating n , we have $\log_b 2b = 4$

$$\text{or } 2b = b^4$$

$$\text{or } b^3 = 2$$

$$\text{or } b = 2^{1/3}$$

$$\Rightarrow \log_2 x \times \log_3 x = \log_2 x + \log_3 x$$

$$\Rightarrow \frac{\log x}{\log 2} \cdot \frac{\log x}{\log 3} = \frac{\log x}{\log 2} + \frac{\log x}{\log 3}$$

$$\Rightarrow \log x = 0$$

$$\text{or } \frac{\log x}{\log 2 \cdot \log 3} = \frac{1}{\log 2} + \frac{1}{\log 3}$$

$$\Rightarrow x = 1 \text{ or } \log x = \log 2 + \log 3$$

$$\Rightarrow x = 1 \text{ or } \log x = \log 6$$

$$\Rightarrow x = 1 \text{ or } x = 6$$

$$12. a^x = b^y = c^z$$

$$\Rightarrow x \log a = y \log b = z \log c$$

$$\therefore \frac{y}{x} = \frac{z}{y} \Rightarrow \frac{\log a}{\log b} = \frac{\log b}{\log c}$$

$$\Rightarrow \log_b a = \log_c b$$

$$13. (a) \frac{\log_n n}{\log_{ab} n} = \frac{\log_n ab}{\log_n a} = \frac{\log_n a + \log_n b}{\log_n a}$$

$$= 1 + \frac{\log_n b}{\log_n a} = 1 + \log_a b$$

$$(b) \frac{\log_a x \log_b x}{\log_a x + \log_b x} = \frac{\frac{1}{\log_x a} \frac{1}{\log_x b}}{\frac{1}{\log_x a} + \frac{1}{\log_x b}} = \frac{\frac{1}{\log_x a \log_x b}}{\frac{\log_x a + \log_x b}{\log_x a \log_x b}}$$

$$= \frac{1}{\log_x a + \log_x b} = \frac{1}{\log_x ab} = \log_{ab} x$$

$$14. \log_{ab} a = 4$$

$$\text{or } \frac{1}{\log_a ab} = 4$$

$$\text{or } \frac{1}{\log_a a + \log_a b} = 4$$

$$\text{or } 1 + \log_a b = \frac{1}{4}$$

$$\text{or } \log_a b = -\frac{3}{4}$$

$$\text{Now, } \log_{ab} (\sqrt[3]{a} / \sqrt{b}) = \frac{\log(\sqrt[3]{a} / \sqrt{b})}{\log ab} = \frac{\frac{1}{3} \log a - \frac{1}{2} \log b}{\log a + \log b}$$

$$= \frac{\frac{1}{3} - \frac{1}{2} \log a}{1 + \frac{\log b}{\log a}}$$

$$= \frac{\frac{1}{3} - \frac{1}{2} \log_a b}{1 + \log_a b}$$

$$= \frac{\frac{1}{3} + \frac{1}{2} \cdot \frac{3}{4}}{1 - \frac{3}{4}}$$

$$= \frac{\frac{17}{24}}{\frac{1}{4}} = \frac{17}{6}$$

$$\frac{17}{4}$$

$$15. \log_a (bcd) = \log_a b + \log_a c + \log_a d$$

$$\text{Now } a^x = b^y$$

$$\text{or } x \log a = y \log b$$

$$\text{or } \frac{\log b}{\log a} = \frac{x}{y}$$

$$\text{or } \log_a b = \frac{x}{y}$$

$$\text{Similarly, } \log_a c = \frac{x}{z} \text{ and } \log_a d = \frac{x}{w}$$

$$\Rightarrow \log_a (bcd) = \log_a b + \log_a c + \log_a d$$

$$= x \left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w} \right)$$

$$16. \left(\frac{1}{49} \right)^{1+\log_7 2} + 5^{-\log_{(1/5)}(7)}$$

$$= (7^{-2})^{\log_7 14} + 5^{\log_5 7}$$

$$= 7^{\log_7 14 \cdot 2} + 7$$

$$= 14^{-2} + 7$$

$$= \frac{1}{196} + 7 = \frac{1373}{196}$$

Exercise 1.4

$$1. \log_2 (25^{x+3} - 1) = 2 + \log_2 (5^{x+3} + 1)$$

$$\text{or } \log_2 (25^{x+3} - 1) - \log_2 (5^{x+3} + 1) = 2$$

$$\text{or } \log_2 \frac{25^{x+3} - 1}{5^{x+3} + 1} = 2$$

$$\text{or } \frac{25^{x+3} - 1}{5^{x+3} + 1} = 2^2$$

$$\text{or } y^2 - 1 = 4y + 4$$

$$\text{or } y^2 - 4y - 5 = 0$$

$$\text{or } y = -1, 5$$

$$\Rightarrow 5^{x+3} = 5$$

$$\text{or } x = -2$$

(putting $5^{x+3} = y$)

$$2. \log_4 (2 \times 4^{x-2} - 1) + 4 = 2x$$

$$\text{or } \log_4 (2 \times 4^{x-2} - 1) = 2x - 4$$

$$\text{or } 2 \times 4^{x-2} - 1 = 4^{2x-4}$$

$$\text{or } 2y - 1 = y^2$$

$$\text{or } y^2 - 2y + 1 = 0$$

$$\text{or } y = 1$$

$$\text{or } 4^{x-2} = 1$$

$$\text{or } x = 2$$

(putting $y = 4^{x-2}$)

$$3. \text{ We have } 27^{\log_3 \sqrt[3]{x^2 - 3x + 1}} = \frac{\log_2 (x-1)}{|\log_2 (x-1)|}$$

We must have $x - 1 > 0$ and $x - 1 \neq 1$

$$\therefore x > 1 \text{ and } x \neq 2$$

$$\text{If } 1 < x < 2, \log_2 (x-1) < 0$$

$$\therefore \frac{\log_2 (x-1)}{|\log_2 (x-1)|} = -1$$

This is not possible as L.H.S. > 0 always.

$$\text{If } x > 2, \text{ then } \frac{\log_2 (x-1)}{|\log_2 (x-1)|} = 1$$

$$\therefore \text{Equation reduces to } 27^{\log_3 \sqrt[3]{x^2 - 3x + 1}} = 1$$

$$\therefore x^2 - 3x + 1 = 1$$

$$\Rightarrow x = 0, 3, \text{ but } x \neq 0$$

$$\therefore x = 3 \text{ is the only solution.}$$

4. The given equality is meaningful if

$$x - 1 > 0, x - 3 > 0 \Rightarrow x > 3.$$

The given equality can be written as

$$\frac{\log(x-1)}{\log 4} = \frac{\log(x-3)}{\log 2}$$

$$\text{or } \log(x-1) = 2 \log(x-3) \quad (\log 4 = 2 \log 2)$$

$$\text{or } (x-1) = (x-3)^2$$

$$\text{or } x^2 - 7x + 10 = 0$$

$$\text{or } (x-5)(x-2) = 0$$

$$\text{or } x = 5 \text{ or } 2.$$

$$\text{But } x > 3, \text{ so } x = 5.$$

$$5. (\log_6 9 + \log_6 4) - \frac{\log_3 27}{\log_3 9} = \frac{\log_8 x}{2} - \log_8 x$$

$$\Rightarrow 2 - \frac{3}{2} = -\frac{1}{2} \log_8 x$$

$$\text{or } \frac{1}{2} = -\frac{1}{2} \log_8 x$$

$$\text{or } x = \frac{1}{8}$$

$$6. \log_2(2\sqrt{17-2x}) = 1 + \log_2(x-1)$$

$$\text{or } \log_2\left(\frac{2\sqrt{17-2x}}{x-1}\right) = 1$$

$$\text{or } \left(\frac{2\sqrt{17-2x}}{x-1}\right) = 2$$

$$\text{or } 2\sqrt{17-2x} = 2(x-1)$$

$$\text{or } x^2 - 2x + 1 = 17 - 2x$$

$$\text{or } x^2 = 16$$

$$\Rightarrow x = 4 \text{ (as } x \neq -4)$$

$$7. 3\log_4 x + 2\log_{4x} 4 + 3\log_{16x} 4 = 0$$

$$\text{or } \frac{3}{\log_4 x} + \frac{2}{\log_4 4x} + \frac{3}{\log_4 16x} = 0$$

$$\text{or } \frac{3}{\log_4 x} + \frac{2}{1 + \log_4 x} + \frac{3}{2 + \log_4 x} = 0$$

$$\text{or } \frac{3}{y} + \frac{2}{1+y} + \frac{3}{2+y} = 0 \quad (\text{putting } y = \log_4 x)$$

$$\text{or } 4y^2 + 8y + 3 = 0$$

$$\text{or } (2y+3)(2y+1) = 0$$

$$\text{or } y = \frac{-1}{2}, \frac{-3}{2}$$

$$\therefore x = \frac{1}{2}, \frac{1}{8}$$

$$8. (\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$$

$$\text{or } \frac{\log x}{\log 3} \times \frac{\log 9}{\log 5} - 2\log_x 5 + \log_3 2 = 3 + \log_3 2$$

$$\text{or } 2\log_3 x - 2\log_x 5 + \log_3 2 = 3 + \log_3 2$$

$$\text{or } 2t^2 - 2 = 3t \quad (\text{putting } \log_3 x = t)$$

$$\text{or } 2t^2 - 3t - 2 = 0$$

$$\text{or } (2t+1)(t-2) = 0$$

$$\Rightarrow t = -1/2 \text{ or } t = 2$$

$$\Rightarrow \log_3 x = -1/2 \text{ or } \log_3 x = 2$$

$$\Rightarrow x = 1/\sqrt{3}, x = 25$$

$$9. \text{ Let } (x^{\log_{10} 3}) = (3^{\log_{10} x}) = t$$

Therefore, the given equation is

$$t^2 - t - 2 = 0 \text{ or } (t-2)(t+1) = 0$$

$$\Rightarrow t = 2$$

(as $t = -1$ is not possible)

$$\Rightarrow (3^{\log_{10} x}) = 2$$

$$\text{or } \log_{10} x = \log_3 2$$

$$\text{or } x = 10^{\log_3 2}$$

$$10. x^{\log_4 x} = 2^{3(\log_4 x + 3)}$$

Taking log on both sides to the base 2, we get

$$(\log_4 x)(\log_2 x) = (\log_4 x + 3) \log_2 8$$

Putting $\log_2 x = t$, we get

$$\frac{1}{2}t^2 = \left(\frac{1}{2}t + 3\right)3$$

$$\text{or } t^2 = 3t + 18$$

$$\text{or } t = 6, -3$$

$$\text{or } x = 2^6, 2^{-3}$$

$$\text{or } x = 64, 1/8$$

$$11. 2\log_{2+\sqrt{3}}(\sqrt{x^2+1}+x) + \log_{2-\sqrt{3}}(\sqrt{x^2+1}-x) = 3$$

$$\Rightarrow 2\log_{2+\sqrt{3}}(\sqrt{x^2+1}+x) + \log_{(2+\sqrt{3})^{-1}}(\sqrt{x^2+1}-x) = 3$$

$$\Rightarrow 2\log_{2+\sqrt{3}}(\sqrt{x^2+1}+x) - \log_{2+\sqrt{3}}(\sqrt{x^2+1}-x) = 3$$

$$\Rightarrow \log_{2+\sqrt{3}} \frac{(\sqrt{x^2+1}+x)^2}{(\sqrt{x^2+1}-x)} = 3$$

$$\Rightarrow \log_{2+\sqrt{3}}(\sqrt{x^2+1}+x)^3 = 3$$

$$\Rightarrow (\sqrt{x^2+1}+x)^3 = (2+\sqrt{3})^3$$

$$\Rightarrow \sqrt{x^2+1}+x = 2+\sqrt{3}$$

$$\Rightarrow x = \sqrt{3}$$

$$12. \text{ We have } x^{\log_{\sqrt{x}} 2x} = 4, x > 0, x \neq 1$$

$$\Rightarrow x^{2\log_x 2x} = 4$$

$$\Rightarrow x^{\log_x 4x^2} = 4$$

$$\Rightarrow 4x^2 = 4$$

$$\Rightarrow x = \pm 1, \text{ which is not possible.}$$

Hence, equation has no solution.

Exercise 1.5

$$1. \log_3 |x| > 2$$

$$\text{or } |x| > 3^2$$

$$\Rightarrow x < -9 \text{ or } x > 9$$

$$1. \log_2 \frac{x-4}{2x+5} < 1$$

$$\Rightarrow 0 < \frac{x-4}{2x+5} < 2^1$$

$$\Rightarrow \frac{x-4}{2x+5} > 0 \text{ and } \frac{x-4}{2x+5} < 2$$

$$\Rightarrow \frac{x-4}{2x+5} > 0 \text{ and } \frac{x-4}{2x+5} - 2 < 0$$

$$\Rightarrow \frac{x-4}{2x+5} > 0 \text{ and } \frac{-3x-14}{2x+5} < 0$$

$$\Rightarrow \frac{x-4}{2x+5} > 0 \text{ and } \frac{3x+14}{2x+5} > 0$$

$$\Rightarrow x < -5/2 \text{ or } x > 4 \text{ and } x < -14/3 \text{ or } x > -5/2$$

$$\Rightarrow x \in (-\infty, -14/3) \cup (4, \infty)$$

$$3. \log_{10}(x^2 - 2x - 2) \leq 0$$

$$\Rightarrow 0 < x^2 - 2x - 2 \leq 10^0$$

$$\Rightarrow x^2 - 2x - 2 > 0 \text{ and } x^2 - 2x - 3 \leq 0$$

$$\Rightarrow x < 1 - \sqrt{3} \text{ or } x > 1 + \sqrt{3}$$

$$\text{and } (x-3)(x+1) \leq 0$$

$$\Rightarrow x \in [-1, 3]$$

From (i) and (ii)

$$x \in [-1, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, 3]$$

$$4. \log_{10} x^2 \geq 0$$

$$\Rightarrow \log_{10} x^2 \geq \log_{10} 1$$

$$\Rightarrow x^2 \geq 1$$

$$\Rightarrow x \geq 1$$

$$\text{or } x \leq -1.$$

$$5. 2^{\log_2(x-1)} > x+5$$

$$\text{or } x-1 > x+5$$

$$\text{or } -1 > 5, \text{ which is not possible.}$$

$$6. |4-5x| > 2^2 = 4$$

$$\text{or } \left| \frac{5x}{4} - 1 \right| > 1$$

$$\Rightarrow \frac{5x}{4} - 1 > 1$$

$$\text{or } \frac{5x}{4} - 1 < -1$$

$$\therefore x > \frac{8}{5} \text{ or } x < 0$$

$$\Rightarrow x \in (-\infty, 0) \cup \left(\frac{8}{5}, +\infty \right)$$

$$7. \frac{x+2}{x} \geq (0.2)^1$$

$$\Rightarrow \frac{x+2}{x} \geq \frac{1}{5}$$

Multiplying by $5x^2$ (positive), we get

$$5x(x+2) \geq x^2 \text{ or } 4x^2 + 10x \geq 0$$

$$\Rightarrow x \geq 0 \text{ or } x \leq -\frac{5}{2}$$

$$\text{Also, } \frac{x+2}{x} > 0$$

$$\Rightarrow x(x+2) > 0$$

$$\Rightarrow x < -2 \text{ or } x > 0$$

Therefore, the solution set is $\left(-\infty, -\frac{5}{2}\right] \cup (0, +\infty)$.

$$8. x^2 - 6x + 12 \leq \left(\frac{1}{2}\right)^{-2}$$

$$\text{or } x^2 - 6x + 8 \leq 0$$

$$\text{or } (x-2)(x-4) \leq 0$$

$$\Rightarrow x \in [2, 4]$$

Also, $x^2 - 6x + 12 > 0$ or $(x-3)^2 + 3 > 0$ which is true for any real x . Hence, $x \in [2, 4]$.

$$9. (0.5)^{\log_3 \log_{1/5} \left(x^2 - \frac{4}{5}\right)} > 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^{\log_3 \log_{1/5} \left(x^2 - \frac{4}{5}\right)} > \left(\frac{1}{2}\right)^0$$

$$\Rightarrow \log_3 \log_{1/5} \left(x^2 - \frac{4}{5}\right) < 0$$

$$\Rightarrow 0 < \log_{1/5} \left(x^2 - \frac{4}{5}\right) < 1$$

$$\Rightarrow 1 > x^2 - \frac{4}{5} > \frac{1}{5}$$

$$\Rightarrow 1 < x^2 < \frac{9}{5}$$

$$\Rightarrow x \in \left(-\frac{3}{\sqrt{5}}, -1\right) \cup \left(1, \frac{3}{\sqrt{5}}\right)$$

$$10. f(x) = \sqrt{\log_{1/2} \frac{x-1}{x+5}}$$

It is defined if $\log_{1/2} \frac{x-1}{x+5} \geq 0$

$$\Rightarrow 0 < \frac{x-1}{x+5} \leq 1$$

$$\text{When } \frac{x-1}{x+5} > 0, x \in (-\infty, -5) \cup (1, \infty) \quad \dots(i)$$

$$\text{When } \frac{x-1}{x+5} \leq 1$$

$$\Rightarrow \frac{x-1}{x+5} - 1 \leq 0$$

$$\Rightarrow \frac{-6}{x+5} \leq 0$$

$$\Rightarrow x > -5 \quad \dots(ii)$$

From (i) and (ii), $x \in (1, \infty)$.

11. The left-hand side of the inequality is defined if $1-x > 0$, $x-2 > 0$, $1-x \neq 1$. Obviously, there is no single value for which these inequalities are satisfied. Thus, the set of its solutions is empty.

$$12. (x+2)(x+4) > 0, x+2 > 0$$

$$\Rightarrow x > -2$$

Given inequality is

$$\log_3 (x+2)(x+4) - \log_3 (x+2) < \frac{(\log 7)/2}{(\log 3)/2}$$

S.6 Trigonometry

$$\Rightarrow \log_3(x+4) < \log_3 7$$

$$\Rightarrow x+4 < 7 \text{ or } x < 3$$

$$\Rightarrow x \in (-2, 3)$$

13. Given $\log_3(x^2 - 1) \leq 0$

If $x > 1$

$$\Rightarrow 0 < x^2 - 1 \leq 1$$

$$\Rightarrow 1 < x^2 \leq 2$$

$$\Rightarrow x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}]$$

$$\Rightarrow x \in (1, \sqrt{2}]$$

If $0 < x < 1$

$$\Rightarrow x^2 - 1 \geq 1$$

$$\Rightarrow x^2 \geq 2$$

$$\Rightarrow x \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$$

$$\Rightarrow x = \phi$$

Thus, $x \in (1, \sqrt{2}]$

Exercise 1.6

1. (a) $N = 3^{40}$

$$\therefore \log_{10} N = 40 \log_{10} 3 = 40 \times 0.477 = 19.08$$

So, number of digits in N is 20.

(b) $N = 2^{32} \times 5^{25} = 2^7(2 \times 5)^{25} = 2^7 \times 10^{25}$

$$\therefore \log_{10} N = 25 + 7 \log_{10} 2$$

$$= 25 + 7 \times 0.3010$$

$$= 25 + 2.107$$

$$= 27.107$$

So, number of digits in N is 28.

(c) $\log_{10} 24^{24} = 24 (\log_{10} (8 \times 3))$

$$= 24 [3 \log_{10} 2 + \log_{10} 3]$$

$$= 24 [3 \times 0.3010 + 0.477]$$

$$= 24 (1.38)$$

$$= 33.12$$

So, number of digits in N is 34.

2. According to the given information,

$$\log a = 5 + p$$

$$\log b = -3 + q$$

$$\log c = 2 + r$$

where p, q and r are mantissas.

$$\therefore p, q, r \in [0, 1)$$

Adding the above equations, we get

$$\log(abc) = 4 + (p + q + r)$$

$$(p + q + r) \in [0, 3)$$

$$\therefore \log(abc) \in [4, 7)$$

$$\Rightarrow \log N \in [4, 7)$$

$$\therefore \text{Maximum possible characteristic of } \log N = 6$$

$$\therefore \text{Maximum number of digits in } N = 6 + 1 = 7$$

3. $N = \frac{ab^2}{c}$

$$\therefore \log_{10} N = \log_{10} a + 2 \log_{10} b - \log_{10} c = 10.28$$

So, characteristic of N is 10.

So, number of digits before decimal is 11.

4. Principal amount = Rs 10,000

Rate of interest = 6%

So, amount after one year = Rs 1.06×10000

Amount after two years = Rs $1.06 \times 1.06 \times 10000$
 $= \text{Rs } (1.06)^2 \times 10000$

and so on

So, amount accumulated after n years = ₹ $(1.06)^n \times 10000$.

Now, $(1.06)^n \times 10000 = 20000$

$$\therefore (1.06)^n = 2$$

$$\Rightarrow n \log_{10} 1.06 = \log_{10} 2$$

$$\Rightarrow n = \frac{\log_{10} 2}{\log_{10} 1.06} = \frac{0.30103}{0.025306}$$

$$\Rightarrow \log n = \log 0.30103 - \log 0.025306$$

$$= -0.5213 - (-1.5968)$$

$$= 1.0755$$

$$= 11.89$$

So, it will take approximately 12 years.

5. Initially bacteria count is 10000.

After 30 min. it doubles i.e., 2×10000 .

After 60 min. it is $2^2 \times 10000$.

So, after t minutes, bacteria count will be $N = 10000 \times 2^{\frac{t}{30}}$

For $N = 100000$, we have

$$100000 = 10000 \times 2^{\frac{t}{30}}$$

$$\therefore 2^{\frac{t}{30}} = 10$$

$$\frac{t}{30} \log_{10} 2 = 1$$

$$\Rightarrow t = \frac{30}{\log_{10} 2} = \frac{30}{0.301} = 99.67$$

So, it will take approximately 100 minutes.

6. $M_A = \log_{10} \frac{I_A}{S}$

$$\therefore 8.3 = \log_{10} \frac{I_A}{S}$$

Now $M_B = \log_{10} \frac{I_B}{S}$

Where $I_B = 4I_A$.

$$\therefore M_B = \log_{10} \frac{4I_A}{S}$$

$$= \log_{10} 4 + \log_{10} \frac{I_A}{S}$$

$$= 0.6020 + 8.3$$

$$= 8.9020$$

So, magnitude of earthquake in city B is 8.9020.

Exercises

Single Correct Answer Type

1. (2) Let $\log_4 18 = p/q$, where $p, q \in I$

$$\Rightarrow \log_4 9 + \log_4 2 = \frac{p}{q}$$

$$\Rightarrow \frac{1}{2} \times 2 \log_2 3 + \frac{1}{2} = \frac{p}{q}$$

$$\Rightarrow \log_2 3 = \frac{p}{q} - \frac{1}{2} = \frac{m}{n} \text{ (say)}$$

where $m, n \in I$ and $n \neq 0$

$$\Rightarrow 3 = (2)^{m/n}$$

or $3^n = 2^m$ (possible only when $m = n = 0$ which is not true)

Hence, $\log_4 18$ is an irrational number.

$$(2) N = \log_{10} 64 + \log_{10} 31 = \log_{10} 1984. \text{ Therefore,}$$

$$3 < N < 4 \Rightarrow 7$$

$$(3) \text{ Let } x = 2000^{2000}$$

$$\log x = 2000 \log_{10}(2000)$$

$$= 2000 (\log_{10} 2 + 3)$$

$$= 2000 (3.3010) = 6602$$

Therefore, the number of digits is 6603.

$$(2) (21.4)^a = 100$$

$$\Rightarrow a \log(21.4) = 2$$

$$\therefore \log(21.4) = 2/a$$

From $(0.00214)^b = 100$, we get

$$b(\log 0.00214) = 2$$

$$\text{or } b \log(21.4 \times 10^{-4}) = 2$$

$$\text{or } b = \frac{2}{\log 21.4 - 4} = \frac{2}{\frac{2}{a} - 4} = \frac{1}{\frac{1}{a} - 2} = \frac{a}{1 - 2a}$$

$$\Rightarrow \frac{1}{b} = \frac{1 - 2a}{a}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = 2$$

(2) $\log ab$ is defined if $ab > 0$ or a and b have the same sign.

Case (i): $a, b > 0$

$$\Rightarrow \log ab - \log|b| = \log a + \log b - \log b = \log a \quad \dots(i)$$

Case (ii): $a, b < 0$

$$\Rightarrow \log ab - \log|b| = \log(-a) + \log(-b) - \log(-b) = \log(-a) \quad \dots(ii)$$

From Eqs. (i) and (ii), we have

$$\log ab - \log|b| = \log|a|.$$

$$(1) \text{ Let } a = x - 1, b = x, c = x + 1$$

$$\text{Now } \log(1 + ac) = \log[1 + (x - 1)(x + 1)]$$

$$= \log x^2 = 2 \log x = 2 \log b$$

$$\Rightarrow K = \log b$$

$$(3) \frac{a + \log_4 3}{a + \log_2 3} = \frac{a + \log_8 3}{a + \log_4 3} = \frac{\log_4 3 - \log_8 3}{\log_2 3 - \log_4 3} = \frac{1}{3}$$

$$(1) \log(p + q) = \log p + \log q$$

$$\Rightarrow p + q = pq$$

$$\Rightarrow pq - p - q = 0$$

$$\Rightarrow pq - p - q + 1 = 1$$

$$\Rightarrow (p - 1)(q - 1) = 1$$

$$\Rightarrow \log(p - 1) + \log(q - 1) = \log 1 = 0$$

$$(4) \frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \frac{(\log_3 2)^2}{(1 + \log_3 2)^2} = \frac{(1 + \log_3 2)^2}{(1 + \log_3 2)^2} = 1$$

$$(4) \text{ Here, } 5 = 4^a \text{ and } 6 = 5^b.$$

$$\text{Let } \log_3 2 = x, \text{ then } 2 = 3^x.$$

$$\text{Now, } 6 = 5^b = (4^a)^b = 4^{ab} \text{ or } 3 = 2^{2ab - 1}$$

$$\therefore 2 = (2^{2ab - 1})^x = 2^{x(2ab - 1)}$$

$$\Rightarrow x(2ab - 1) = 1$$

$$(2) \log_{0.72}(9.6) = \log_{\frac{72}{100}}\left(\frac{96}{10}\right) = \frac{(\log_{10} 96 - \log_{10} 10)}{(\log_{10} 72 - \log_{10} 100)} = \frac{\log_{10}(2^5 \times 3) - 1}{\log_{10}(2^3 \times 3^2) - 2} = \frac{5a + b - 1}{3a + 2b - 2}$$

$$12. (1) N = 50 \log_{10} N$$

$$\Rightarrow \log_{10} N = \frac{N}{50}$$

$$\Rightarrow N = (10)^{N/50}$$

$$\Rightarrow N^{1/N} = (10)^{1/50} = (10)^{2/100} = (100)^{1/100}$$

$$\therefore N = 100$$

$$13. (1) \text{ Let } \log_2 12 = a, \text{ then}$$

$$\frac{1}{\log_{96} 2} = \log_2 96 = \log_2 2^3 \times 12 = 3 + a$$

$$\log_2 24 = 1 + a$$

$$\Rightarrow \log_2 192 = \log_2 (16 \times 12) = 4 + a$$

$$\text{and } \frac{1}{\log_{12} 2} = \log_2 12 = a.$$

Therefore, the given expression is:

$$(1 + a)(3 + a) - (4 + a)a = 3$$

$$14. (2) \log_{(x-12)} x = 2$$

$$x = (x - 12)^2$$

$$x^2 - 25x + 144 = 0$$

$$x = 9, 16$$

$$\text{but } x \neq 9, x = 16$$

$$15. (4) f(x_1) + f(x_2) = \log \left(\frac{1 + x_1}{1 - x_1} \cdot \frac{1 + x_2}{1 - x_2} \right)$$

$$= \log \left(\frac{1 + x_1 x_2 + x_1 + x_2}{1 + x_1 x_2 - x_1 - x_2} \right)$$

$$= \log \left(\frac{1 + \frac{x_1 + x_2}{1 + x_1 x_2}}{1 - \frac{x_1 + x_2}{1 + x_1 x_2}} \right) = f \left(\frac{x_1 + x_2}{1 + x_1 x_2} \right)$$

$$16. (1) \text{ Given } 4 \log_a a + 5 \log_a b = 0$$

$$\Rightarrow \log_a b = -4/5 \quad \dots(i)$$

$$\text{Now } \log_a (a^5 b^4) = 5 + 4 \log_a b = 5 + 4 \left(-\frac{4}{5} \right)$$

$$= 5 - \frac{16}{5} = \frac{9}{5}$$

$$17. (1) \text{ Using property,}$$

$$3^{\log_4 5} = 5^{\log_4 3}$$

$$\therefore 3^{\log_4 5} - 5^{\log_4 3} = 0$$

$$18. (3) \text{ Taking log, we have}$$

$$(x + y) \log 2 = y (\log 2 + \log 3)$$

$$\therefore x \log 2 = y \log 3$$

$$\text{or } \frac{x}{\log 3} = \frac{y}{\log 2} = \frac{x - y}{\log 3 - \log 2} = \lambda \text{ (say)} \quad \dots(i)$$

$$\text{Also } (x - 1) \log 3 = (y + 1) \log 2$$

$$\text{or } x \log 3 - y \log 2 = \log 3 + \log 2$$

Using Eq. (i), we get

$$\lambda [(\log 3)^2 - (\log 2)^2] = \log 3 + \log 2$$

$$\lambda = \frac{1}{\log 3 - \log 2}$$

$$\therefore \frac{1}{\lambda} = \log 3 - \log 2 = \log \frac{3}{2}$$

S.8 Trigonometry

19. (4) $3^{-2} 3^{\log_{\sqrt{5}} x} = 3^{-2}$
 $\Rightarrow 3^{\log_{\sqrt{5}} x} = 1$
 $\Rightarrow \log_{\sqrt{5}} x = 0$
 $\Rightarrow x = 1$
20. (4) Using $a^{\log_a N} = N$ repeatedly, we get
 $(\sqrt[3]{5})^{\log_5 \left(\frac{x}{2}\right)} = 3$
 $\Rightarrow 5^{\log_5 \left(\frac{x}{2}\right)^{1/3}} = 3$
 $\Rightarrow \left(\frac{x}{2}\right)^{1/3} = 3$
 $\Rightarrow \frac{x}{2} = 27$
 $\Rightarrow x = 54$
21. (2) $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$
or $\sqrt{\log_2 x} - 0.5 = 0.5 \log_2 x$
 $\Rightarrow y - 0.5 = 0.5 y^2$
 $\Rightarrow y^2 - 2y + 1 = 0$
 $\Rightarrow y = 1$
 $\Rightarrow \log_2 x = 1$
 $\Rightarrow x = 2$
22. (1) Let $t = \log_y x$, $x, y > 0$, and $\neq 1$, then $t + \frac{1}{t} = 2$ or $(t-1)^2 = 0$
 $\therefore t = \log_y x = 1$, i.e., $x = y$. We get
 $x^2 + x - 12 = 0$ $x = -4, 3$.
 $x = 3$ only (-4 rejected)
 $\therefore y = 3$,
 $\therefore x = 9$
23. (3) (4) $10^{\log_5 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$
or $(4)^{\frac{1}{2} \log_3 3} + (9)^{2 \log_2 2} = (10)^{\log_x 83}$
or $2 + 81 = (10)^{\log_x 83}$
or $83 = (10)^{\log_x 83}$
 $\Rightarrow x = 10$
24. (3) $(x+1)^{\log_{10}(x+1)} = 100(x+1)$
 $\Rightarrow \log_{10}(x+1)^{\log_{10}(x+1)} = \log_{10}(100(x+1))$
 $\Rightarrow \log_{10}(x+1) \log_{10}(x+1) = 2 + \log_{10}(x+1)$
Let $\log_{10}(x+1) = y$
 $\Rightarrow y^2 - y - 2 = 0$
 $\Rightarrow y = 2$ or -1
 $\Rightarrow \log_{10}(x+1) = 2, -1$
 $\Rightarrow x+1 = 100, 1/10$
 $\Rightarrow x = 99$ or $-9/10$
25. (4) $\frac{10}{3} = 3 + \frac{1}{3}$. The given equation is of the form
 $p + \frac{1}{p} = 3 + \frac{1}{3} = q + \frac{1}{q}$, where $p \neq q$ as $x \neq y$
 $\Rightarrow \log_2 x = 3, \log_2 y = 1/3$
 $\Rightarrow x = 2^3, y = 2^{1/3}$
 $\therefore x + y = 8 + 2^{1/3}$
26. (4) R.H.S. $= x = [\log_{10} 5 - \log_{10} 10] = x \log_{10} \frac{5}{10} = \log_{10} \frac{1}{2^x}$
 $\Rightarrow \frac{1}{2^x + x - 1} = \frac{1}{2^x}$
Therefore $x - 1 = 0$ or $x = 1$.
27. (2) We must have $x - 1 > 0$
 $\Rightarrow x > 1$
and $5 + 4 \log_3 (x-1) > 0$... (i)
or $4 \log_3 (x-1) > -5$
or $\log_3 (x-1) > -\frac{5}{4}$
or $x-1 > 3^{-5/4}$
or $x > 1 + 3^{-5/4}$... (ii)
From Eqs. (i) and (ii), we get $x > 1 + 3^{-5/4}$.
 $\therefore 5 + 4 \log_3 (x-1) = 9$
 $\Rightarrow 4 \log_3 (x-1) = 4$
or $\log_3 (x-1) = 1$
or $x-1 = 3$ or $x = 4$
28. (4) $2x^{\log_4 3} + 3^{\log_4 x} = 27$
 $\Rightarrow 2 \times 3^{\log_4 x} + 3^{\log_4 x} = 27$
 $\Rightarrow 3^{\log_4 x} = 9 = 3^2$
 $\Rightarrow \log_4 x = 2$
 $\therefore x = 4^2 = 16$.
29. (4) $\log_4 (3-x) + \log_{0.25} (3+x) = \log_4 (1-x) + \log_{0.25} (2x+1)$
 $\Rightarrow \log_4 (3-x) - \log_4 (3+x) = \log_4 (1-x) - \log_4 (2x+1)$
 $\Rightarrow \log_4 (3-x) + \log_4 (2x+1) = \log_4 (1-x) + \log_4 (3+x)$
 $\Rightarrow (3-x)(2x+1) = (1-x)(3+x)$
 $\Rightarrow 3 + 5x - 2x^2 = 3 - 2x - x^2$
 $\Rightarrow x^2 - 7x = 0$
 $\Rightarrow x = 0, 7$
Only $x = 0$ is the solution and $x = 7$ is to be rejected.
30. (3) $\frac{2 \log_5 (bx+28)}{\log_5 (1/5)^2} = -\log_5 (12-4x-x^2)$
 $\Rightarrow bx+28 = 12-4x-x^2$
or $x^2 + (b+4)x + 16 = 0$
For coincident roots,
 $D = 0$
 $\Rightarrow (b+4)^2 = 4(16)$
 $\Rightarrow b+4 = \pm 8$
31. (2) $2^{2y} - 2^y + 2^x(1-2^x) = 0$
Putting $2^y = t$, we get
 $t^2 - t + 2^x(1-2^x) = 0$, where $t_1 = 2^{y_1}$ and $t_2 = 2^{y_2}$
 $t_1 t_2 = 2^x(1-2^x)$
 $2^{y_1+y_2} = 2^x(1-2^x)$
 $y_1 + y_2 = x + \log_2(1-2^x)$
32. (1) $x^{\log_5 (x+3)^2} = 16$, where $x > 0, x \neq 1$
 $\Rightarrow (x+3)^2 = 16$
or $x = 1, -7$
Therefore, both values are not possible.
33. (4) Let $\log_8 x = y$, then the given equation reduces to
 $(1-2y)/y^2 = 3$.
 $\Rightarrow 3y^2 + 2y - 1 = 0$

$$(3y-1)(y+1)=0$$

$$\Rightarrow \log_8 x = y = 1/3, -1$$

$$\Rightarrow x = 2, 1/8$$

34. (4) Given equation can be written as

$$(a^{\log_2 x})^2 = 5 + 4a^{\log_2 x}$$

Let $a^{\log_2 x} = t$. Then the given equation becomes

$$t^2 - 4t - 5 = 0$$

$$\Rightarrow (t-5)(t+1) = 0$$

$$\Rightarrow t = 5 \text{ or } t = -1 \text{ (rejected)}$$

$$\therefore a^{\log_2 x} = 5$$

$$\Rightarrow x^{\log_2 a} = 5$$

$$\Rightarrow x = 5^{\log_a 2}$$

$$35. (2) \frac{\log x}{\log 3 + (1/2)\log x} + \frac{(1/2)\log x}{\log 3 + \log x} = 0$$

$$\Rightarrow \frac{\log_3 x}{1 + (1/2)\log_3 x} + \frac{1}{2} \frac{\log_3 x}{(1 + \log_3 x)} = 0$$

$$\text{Let } \log_3 x = y, \text{ we get } \frac{y}{1 + (y/2)} + \frac{y}{2(1+y)} = 0$$

$$\text{or } y \left(\frac{2}{2+y} + \frac{1}{2(1+y)} \right) = 0$$

$$\text{or } y[4 + 4y + 2 + y] = 0$$

$$\Rightarrow y = 0 \text{ or } y = -6/5$$

$$\Rightarrow \log_3 x = 0 \text{ or } \log_3 x = -6/5$$

$$\Rightarrow x = 1 \text{ or } x = 3^{-6/5}$$

36. (2) Taking log of both sides with base 3, we have

$$(\log_3 x^2 + (\log_3 x)^2 - 10)(\log_3 x) = -2 \log_3 x$$

$$\Rightarrow \log_3 x = 0 \text{ or } 2 \log_3 x + (\log_3 x)^2 - 8 = 0$$

$$\Rightarrow x = 1, \text{ or } \log_3 x = 2, -4.$$

$$\text{Hence, } x = 1, 3^2, 3^{-4}$$

$$37. (2) \text{ We have } \log_2(x^2 - x) \log_2 \left(\frac{x-1}{x} \right) + (\log_2 x)^2 = 4$$

$$\Rightarrow [\log(x-1) + \log x][\log(x-1) - \log x] + (\log_2 x)^2 = 4$$

$$\Rightarrow [\log_2(x-1)]^2 - (\log_2 x)^2 + (\log_2 x)^2 - 4 = 0$$

$$\Rightarrow [\log_2(x-1)]^2 = 4$$

$$\Rightarrow \log_2(x-1) = \pm 2$$

$$\Rightarrow x-1 = 4 \text{ or } \frac{1}{4}$$

$$\Rightarrow x = 5 \text{ or } \frac{5}{4}$$

$$38. (4) \log_3(\log_2 x) + \log_{1/3}(\log_{1/2} y) = 1$$

$$\text{or } \log_3(\log_2 x) - \log_3(\log_{1/2} y) = 1$$

$$\text{or } \log_3(\log_2(4/y^2)) - \log_3(\log_{1/2} y) = 1$$

$$\text{or } \log_2(4/y^2) = 3(\log_{1/2} y)$$

$$\text{or } \log_2(4/y^2) = -3(\log_2 y)$$

$$\text{or } \log_2(4/y^2) + (\log_2 y^3) = 0$$

$$\text{or } 4y = 1$$

$$\text{or } y = 1/4$$

$$\Rightarrow x = 64$$

$$39. (2) e^2 \cdot x^{\ln x} = x^3$$

Taking log on both sides, we get

$$\ln(e^2 \cdot x^{\ln x}) = \ln(x^3)$$

$$\Rightarrow (\ln x)^2 - 3 \ln x + 2 = 0$$

$$\Rightarrow (\ln x - 2)(\ln x - 1) = 0$$

$$\text{If } \ln x = 2 \Rightarrow x = e^2$$

$$\text{If } \ln x = 1 \Rightarrow x = e$$

Since $x_1 > x_2$, we get $x_1 = e^2$ and $x_2 = e$

$$\Rightarrow x_2^2 = x_1$$

$$40. (1) \log_{16} x = \frac{1 \pm \sqrt{1 - 4 \log_{16} k}}{2}$$

For exactly one solution, $4 \log_{16} k = 1$.

$$\Rightarrow k^4 = 16, \text{ i.e., } k = 2, -2.$$

$$41. (2) x^{\log_5 x} > 5$$

Taking logarithm with base 5, we have

$$(\log_5 x)(\log_5 x) > 1$$

$$\text{or } (\log_5 x - 1)(\log_5 x + 1) > 0$$

$$\text{or } \log_5 x > 1 \text{ or } \log_5 x < -1$$

$$\Rightarrow x > 5 \text{ or } x < 1/5$$

Also we must have $x > 0$. Thus,

$$x \in (0, 1/5) \cup (5, \infty).$$

$$42. (1) 2 + \log_2 \sqrt{x+1} > 1 - \log_{1/2} \sqrt{4-x^2}$$

$$\Rightarrow 1 + \log_2 \sqrt{x+1} - \log_2 \sqrt{4-x^2} > 0$$

$$\text{or } \log_2 2 + \log_2 \sqrt{x+1} - \log_2 \sqrt{4-x^2} > 0$$

$$\text{or } \log_2 \frac{2\sqrt{x+1}}{\sqrt{4-x^2}} > 0 \text{ or } \frac{2\sqrt{x+1}}{\sqrt{4-x^2}} > 1$$

$$\therefore 4(x+1) > 4-x^2$$

$$\therefore x^2 + 4x > 0$$

$$x < -4 \text{ or } x > 0$$

...(i)

$$\text{Also, } x+1 > 0 \text{ or } x > -1$$

$$\text{and } 4-x^2 > 0$$

$$\text{or } x \in (-2, 2)$$

$$\therefore x \in (0, 2)$$

$$\Rightarrow x = 1, \text{ as } x \in N$$

$$43. (2) (\log_{0.6} (0.6)^3) \log_5 (5-2x) \leq 0$$

$$\Rightarrow 5-2x \leq 1$$

$$\Rightarrow x \geq 2$$

...(i)

$$\text{Also, } 5-2x > 0$$

...(ii)

From Eqs. (i) and (ii), we have $x \in [2, 2.5)$.

44. (4) Put $2^x = t$. Then $t > 0$. The given inequality becomes

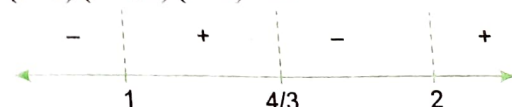
$$\frac{1}{t-1} > \frac{2}{2-t}$$

$$\Rightarrow \frac{1}{t-1} - \frac{2}{2-t} > 0$$

$$\Rightarrow \frac{2-t-2t+2}{(t-1)(2-t)} > 0$$

$$\Rightarrow \frac{4-3t}{(t-1)(2-t)} > 0$$

$$\Rightarrow (t-1)(t-4/3)(t-2) > 0.$$



From above sign scheme, we get

$$1 < t < 4/3 \text{ or } t > 2.$$

S.10 Trigonometry

$$\Rightarrow 1 < 2^x < 4/3 \text{ or } 2^x > 2$$

$$\Rightarrow 0 < x < \log_2(4/3) \text{ or } x > 1$$

45. (3) Given $\log_2 x + \log_2 y \geq 6$

$$\Rightarrow \log_2(xy) \geq 6$$

$$\Rightarrow xy \geq 64$$

Also to define $\log_2 x$ and $\log_2 y$, we have

$$x > 0, y > 0.$$

Since A.M. $\geq$ G.M., we have

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$\text{or } x+y \geq 2\sqrt{xy} \geq 16$$

46. (3) Given inequality is defined if $x > 2/5$; $x \neq 1$

Case I: If $x > 1$

$$\Rightarrow \frac{5}{2} - \frac{1}{x} > x$$

$$\text{or } x + \frac{1}{x} < \frac{5}{2}$$

$$\text{or } 2(x^2 + 1) < 5x$$

$$\text{or } 2x^2 - 5x + 2 < 0$$

$$\text{or } 2x^2 - 4x - x + 2 < 0$$

$$\text{or } (x-2)(2x-1) < 0$$

$$\Rightarrow x \in (1, 2)$$

...(i)

Case II: $\frac{2}{5} < x < 1$, then $(x-2)(2x-1) > 0$

$$\Rightarrow x \in \left(\frac{2}{5}, \frac{1}{2}\right)$$

...(ii)

From Eqs. (i) and (ii), we get $x \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup (1, 2)$.

47. (2) $x^2 - 16 \leq 4x - 11$

$$\Rightarrow x^2 - 4x - 5 \leq 0$$

$$\Rightarrow (x-5)(x+1) \leq 0$$

$$\Rightarrow -1 \leq x \leq 5$$

...(i)

Also $x^2 - 16 > 0$

$$\Rightarrow x < -4 \text{ or } x > 4$$

...(ii)

And $4x - 11 > 0$

$$\Rightarrow x > 11/4$$

...(iii)

From Eqs. (i), (ii) and (iii), we have $x \in (4, 5]$.

48. (4) $\log_{6/5} \left(\log_6 \frac{x^2+x}{x+4} \right) < 0$

$$\Rightarrow \log_6 \frac{x^2+x}{x+4} > 1$$

$$\Rightarrow \frac{x^2+x}{x+4} > 6$$

$$\Rightarrow \frac{x^2+x}{x+4} - 6 > 0$$

$$\Rightarrow \frac{x^2-5x-24}{(x+4)} > 0$$

$$\Rightarrow \frac{(x-8)(x+3)}{(x+4)} > 0$$

From the sign scheme of $\frac{(x-8)(x+3)}{(x+4)}$,

$$x \in (-4, -3) \cup (8, \infty).$$

49. (3) We have, $\log_3(x^2-2) < \log_3\left(\frac{3}{2}|x|-1\right)$

For this to be true, we must have

$$x^2 - 2 > 0, \frac{3}{2}|x| - 1 > 0$$

$$\text{and } x^2 - 2 < \frac{3}{2}|x| - 1$$

$$\Rightarrow x^2 > 2, |x| > \frac{2}{3} \text{ and } 2|x|^2 - 3|x| - 2 < 0$$

$$\Rightarrow |x| > \sqrt{2}, |x| > \frac{2}{3} \text{ and } (2|x|+1)(|x|-2) < 0$$

$$\Rightarrow |x| > \sqrt{2} \text{ and } |x| < 2$$

$$\Rightarrow \sqrt{2} < |x| < 2$$

$$\Rightarrow x \in (-2, -\sqrt{2}) \cup (\sqrt{2}, 2)$$

50. (4) $\log_{(x+1)}(x^2-4) > 1$

We must have $x^2-4 > 0$, $x+1 > 0$ and $x+1 \neq 1$

$$\therefore x \in (2, \infty)$$

...(i)

Case I:

$$x+1 > 1 \Rightarrow x > 0$$

$$\therefore x^2-4 > x+1$$

$$\therefore x^2-x-5 > 0$$

$$\therefore x \in \left(\frac{1+\sqrt{21}}{2}, \infty\right)$$

...(ii)

Case II:

$$(x+1) \in (0, 1)$$

$$\Rightarrow x \in (-1, 0)$$

Clearly, this is not possible as $x \in (2, \infty)$.

Multiple Correct Answers Type

1. (1), (4)

$$\log_{ax} a + \log_x a^2 + \log_{a^2x} a^3 = 0$$

$$\text{or } \frac{1}{\log_a ax} + \frac{2}{\log_a x} + \frac{3}{(\log_a a^2x)} = 0$$

$$\text{or } \frac{1}{1+\log_a x} + \frac{2}{\log_a x} + \frac{3}{(2+\log_a x)} = 0$$

Let $\log_a x = y$, we have $\frac{1}{y+1} + \frac{2}{y} + \frac{3}{2+y} = 0$

$$\text{or } 6y^2 + 11y + 4 = 0$$

$$\text{or } y = \log_a x = -\frac{1}{2}, -\frac{4}{3}$$

$$\Rightarrow x = a^{-4/3}, a^{-1/2}$$

2. (2), (3), (4)

$$2^{x+2} \cdot 5^{6-x} = 2^{x^2} \cdot 5^{x^2}$$

$$\text{or } 5^{6-x-x^2} = 2^{x^2-x-2}$$

$$\text{or } (6-x-x^2) \log_{10} 5 = (x^2-x-2) \log_{10} 2 \text{ (base 10)}$$

$$\text{or } (6-x-x^2) [1 - \log_{10} 2] = (x^2-x-2) \log_{10} 2$$

$$\text{or } 6-x-x^2 = (\log_{10} 2) [(x^2-x-2) - x^2-x+6]$$

$$\text{or } 6 - x - x^2 = (\log_{10} 2) [4 - 2x]$$

$$\text{or } x^2 + x - 6 = 2 (\log_{10} 2) (x - 2)$$

$$\text{or } (x + 3)(x - 2) = (\log_{10} 4)(x - 2)$$

Therefore, either $x = 2$ or $x + 3 = \log_{10} 4$

$$\Rightarrow x = \log_{10} 4 - 3 = \log_{10} \left(\frac{4}{1000} \right)$$

$$\Rightarrow x = -\log_{10} (250)$$

3. (1), (2), (3), (4)

$$\text{Let } \frac{\log_k x}{b-c} = \frac{\log_k y}{c-a} = \frac{\log_k z}{a-b} = p$$

$$\Rightarrow x = k^{p(b-c)}, y = k^{p(c-a)}, z = k^{p(a-b)}$$

$$\Rightarrow xyz = k^{p(b-c)} k^{p(c-a)} k^{p(a-b)}$$

$$= k^{p(b-c) + p(c-a) + p(a-b)} = k^0 = 1$$

$$x^a y^b z^c = k^{pa(b-c)} k^{pb(c-a)} k^{pc(a-b)} = k^0 = 1$$

$$x^{b+c} y^{c+a} z^{a+b} = k^{p(b+c)(b-c)} k^{p(c+a)(c-a)} k^{p(a+b)(a-b)} = k^0 = 1$$

4. (2), (3)

$$\log_k x \cdot \log_5 k = \log_x 5$$

$$\text{or } \frac{\log x}{\log k} \frac{\log k}{\log 5} = \log_x 5$$

$$\text{or } \frac{\log x}{\log 5} = \log_x 5$$

$$\text{or } \log_5 x = \frac{1}{\log_5 x}$$

$$\text{or } (\log_5 x)^2 = 1$$

$$\text{or } \log_5 x = \pm 1$$

$$\Rightarrow x = 5^{\pm 1}$$

$$\Rightarrow x = \frac{1}{5}, 5$$

5. (1), (3), (4)

$$x^{\sqrt{x}} = (\sqrt{x})^x, p, q \in \mathbb{N}$$

$$\Rightarrow \sqrt{x} \log x = x \log \sqrt{x}$$

$$\text{or } \log x \left[\sqrt{x} - \frac{x}{2} \right] = 0$$

$$\Rightarrow \log x = 0 \text{ or } \left[\sqrt{x} - \frac{x}{2} \right] = 0$$

$$\Rightarrow x = 1 \text{ or } 4$$

6. (1), (2), (3)

$$(1) \log_{10} \left(\frac{10}{2} \right) \cdot \log_{10} (10 \times 2) + (\log_{10} 2)^2 \\ = (1 - \log_{10} 2) (1 + \log_{10} 2) + (\log_{10} 2)^2 = 1$$

$$(2) \frac{\log 2^2 \times 3}{\log(48/4)} = 1$$

$$(3) -\log_5 \log_3 9^{1/10} = -\log_5 \log_3 3^{1/5} \\ = -\log_5 (1/5) = 1$$

$$(4) \frac{1}{6} \log_{\sqrt{3}} \left(\frac{64}{27} \right) = \frac{1}{6} \log_{\sqrt{3}} \left(\frac{\sqrt{3}}{2} \right)^{-6} = -1$$

7. (1), (2), (3), (4)

$$\log_a x = b \text{ or } x = a^b$$

(1) For $a = \sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ and $b = \sqrt{2} \notin \mathbb{Q}$; $x = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ which is rational.

(2) For $a = 2 \in \mathbb{Q}$ and $b = \log_2 3 \notin \mathbb{Q}$; $x = 3$ which is rational.

(3) For $a = \sqrt{2}$ and $b = 2$; $x = 2$

(4) The option is obviously correct.

8. (2), (3), (4)

$$\frac{\log_2(x-0.5)}{\log_2(x+1)} = \frac{\log_2(x+1)}{\log_2(x-0.5)}$$

$$\text{or } [\log_2(x+1)]^2 = [\log_2(x-0.5)]^2$$

$$\therefore \log_2(x+1) = \pm \log_2(x-0.5)$$

If $\log_2(x+1) = \log_2(x-0.5)$. Then, $x+1 = x-0.5$, hence no solution

If $\log_2(x+1) = \log_2(x-0.5)^{-1}$. Then,

$$x+1 = \frac{1}{x-(1/2)} = \frac{2}{2x-1}$$

$$\text{or } (x+1)(2x-1) = 2$$

$$\text{or } 2x^2 + x - 3 = 0$$

$$\text{or } 2x^2 + 3x - 2x - 3 = 0$$

$$\text{or } (x-1)(2x+3) = 0$$

$$\Rightarrow x = 1$$

($x = -3/2$ is rejected)

9. (1), (4)

$$\left(\sqrt{1 + \frac{3}{2\log_3 x}} \right) \log_3 x + 1 = 0$$

Let $\log_3 x = y$, we get

$$\left(\sqrt{1 + \frac{3}{2y}} \right) y = -1$$

$$\text{or } \left(1 + \frac{3}{2y} \right) = \frac{1}{y^2}$$

$$\text{or } \frac{2y+3}{2y} = \frac{1}{y^2}$$

$$\text{or } 2y^2 + 3y - 2 = 0 \quad (\because y \neq 0)$$

$$\text{or } 2y^2 + 4y - y - 2 = 0$$

$$\text{or } (y+2)(2y-1) = 0$$

$$y = 1/2 \text{ or } y = -2 \quad (\text{not possible})$$

$$\Rightarrow x = 3^{1/2}$$

10. (1), (2)

$$\log_{1/2}(4-x) \geq \log_{1/2} 2 - \log_{1/2}(x-1)$$

$$\text{or } \log_{1/2}(4-x)(x-1) \geq \log_{1/2} 2$$

$$\text{or } (4-x)(x-1) \leq 2$$

$$\text{or } x^2 - 5x + 6 \geq 0$$

$$\text{or } (x-3)(x-2) \geq 0$$

$$\text{or } x \geq 3 \text{ or } x \leq 2$$

$$\text{But } x \in (1, 4)$$

$$\Rightarrow x \in (1, 2] \cup [3, 4)$$

11. (1), (3)

$$(\log_a x^2) \log_a x = (k-2) \log_a x - k$$

(taking log on base a)

Let $\log_a x = t$, we get

$$2t^2 - (k-2)t + k = 0$$

Putting $D = 0$ (has only one solution), we have

$$(k-2)^2 - 8k = 0$$

$$\text{or } k^2 - 12k + 4 = 0$$

$$\text{or } k = \frac{12 \pm \sqrt{128}}{2}$$

$$\text{or } k = 6 \pm 4\sqrt{2}$$

12. (3), (4)

$$|x-1|^{\log_3 x^2 - 2\log_3 9} = (x-1)^7$$

Since L.H.S. > 0 . So, $x > 1$

$$\therefore (x-1)^{\log_3 x^2 - 2\log_3 9} = (x-1)^7$$

$$\Rightarrow x-1 = 1 \text{ or } \log_3 x^2 - 2\log_3 9 = 7$$

$$\Rightarrow x = 2 \text{ or } 2\log_3 x - 4 \frac{1}{\log_3 x} - 7 = 0$$

$$\Rightarrow x = 2 \text{ or } 2(\log_3 x)^2 - 7\log_3 x - 4 = 0$$

$$\Rightarrow x = 2 \text{ or } \log_3 x = -1/2, 4$$

$$\Rightarrow x = 2 \text{ or } x = 3^{-1/2}, 3^4$$

$$\Rightarrow x = 2, 81$$

($\because x > 1$)

13. (2), (3)

$$\log_e(x^2 + 15a^2) - \log_e(a-2) = \log_e\left(\frac{8ax}{a-2}\right)$$

$$\text{Here } a > 2, \frac{8ax}{a-2} > 0$$

$$\therefore x > 0$$

$$\text{Now } \frac{x^2 + 15a^2}{a-2} = \frac{8ax}{a-2}$$

$$\text{or } x^2 - 8ax + 15a^2 = 0$$

$$\Rightarrow x = 3a, 5a$$

$$\text{Now, } x = 9$$

$$\Rightarrow a = 3, \frac{9}{5}$$

$$\text{But } a > 2$$

$$\therefore a = 3$$

$$\text{For } a = 3, x = 9, 15$$

14. (1), (4)

$$(1) m - n = (\log_2 5)^2 - (\log_2 5 + 2) \\ = (\log_2 5 - 2)(\log_2 5 + 1) > 0$$

$$\therefore m > n$$

$$(2) 2^3 < 10$$

$$\text{or } 2 < 10^{1/3}$$

$$\text{or } \log_{10} 2 < \log_{10} 10^{1/3}$$

$$\therefore m < n$$

$$(3) m = \log_{10} 5 \cdot \log_{10} 20 \\ = (\log_{10} 10 - \log_{10} 2)(\log_{10} 10 + \log_{10} 2) \\ = (1 - \log_{10} 2)(1 + \log_{10} 2) \\ = 1 - (\log_{10} 2)^2 \\ < 1$$

$$\therefore m < n$$

$$(4) m = \log_{1/2} \left(\frac{1}{3}\right) = \log_2 3 \text{ and } n = \log_{1/3} \left(\frac{1}{2}\right) = \log_3 2$$

$$\therefore \frac{m}{n} = \frac{\log_2 3}{\log_3 2} = (\log_2 3)^2 > 1$$

$$\therefore m > n \text{ (as } n > 0)$$

15. (1), (2), (3)

$$(1) \log_{30} 8 = \frac{3\log_{10} 2}{\log_{10} 5 + \log_{10} 3 + \log_{10} 2} \\ = \frac{3(1 - \log_{10} 5)}{\log_{10} 5 + \log_{10} 3 + \log_{10} 2} \\ = \frac{3(1 - \log_{10} 5)}{1 + \log_{10} 3} = \frac{3(1-a)}{1+b}$$

$$(2) \log_{40} 15 = \frac{\log_{10} 15}{\log_{10} 40} \\ = \frac{\log_{10} 3 + \log_{10} 5}{\log_{10} 5 + 3[1 - \log_{10} 5]} = \frac{a+b}{3-2a}$$

$$(3) \log_{243} 32 = \frac{\log_{10} 2}{\log_{10} 3} = \frac{1 - \log_{10} 5}{\log_{10} 3} = \frac{1-a}{b}$$

16. (1), (2)

$$\frac{6 a^{\log_e b} \log_a b \cdot \log_b a}{e^{\log_e a \cdot \log_e b}} = \frac{6 a^{\log_e b} \frac{1}{2} \log_a b \cdot \frac{1}{2} \log_b a}{(e^{\log_e a})^{\log_e b}} \\ = \frac{6}{4} \frac{a^{\log_e b}}{a^{\log_e b}} \\ = \frac{3}{2}$$

17. (2), (3)

$$\sqrt{x^{\log_2 \sqrt{x}}} \geq 2, x > 0$$

$$\Rightarrow x^{\log_2 \sqrt{x}} \geq 4$$

Taking log on both sides with base 2, we get

$$\log_2 x^{\log_2 \sqrt{x}} \geq 2$$

$$\text{or } (\log_2 \sqrt{x})(\log_2 x) \geq 2$$

$$\text{or } (1/2)(\log_2 x)(\log_2 x) \geq 2$$

$$\text{or } (\log_2 x)^2 \geq 4$$

$$\text{or } (\log_2 x - 2)(\log_2 x + 2) \geq 0$$

$$\Rightarrow \log_2 x \leq -2 \text{ or } \log_2 x \geq 2$$

$$\Rightarrow x \leq 1/4 \text{ or } x \geq 4$$

$$\Rightarrow x \in (0, 1/4] \cup [4, \infty)$$

Linked Comprehension Type

1. (3), 2. (2)

$$\log_3 (\log_2 x) + \log_{1/3} (\log_{1/2} y) = 1 \\ \Rightarrow \log_3 (\log_2 x) - \log_3 (-\log_2 y) = 1$$

$$\Rightarrow \log_3 \left(-\frac{\log_2 x}{\log_2 y} \right) = 1$$

$$\Rightarrow -\frac{\log_2 x}{\log_2 y} = 3$$

$$\Rightarrow xy^3 = 1$$

$$\text{Also, } xy^2 = 9$$

$$\Rightarrow y = \frac{1}{9}$$

$$\therefore x = 729$$

4. (2), 4. (4)

$$\text{Let } \log_r x = t$$

$$\text{Then } x = r^t$$

$$\text{Now, } x^{\log_r x} = 2 \text{ becomes}$$

$$x^t = 2$$

$$\Rightarrow x = 2^{1/t}$$

$$\text{And } y^{\log_r y} = 16 \text{ becomes}$$

$$y^{1/t} = 2^4$$

$$\Rightarrow y = 2^{4t}$$

Putting the values of x and y in (1), we get

$$2^{1/t} = 2^{4t^2}$$

$$\Rightarrow 4t^3 = 1$$

$$\therefore t = \left(\frac{1}{4}\right)^{1/3}$$

$$\text{Using (4) and (2), we get } x = (2)^{(4)^{1/3}} = 2^{\sqrt[3]{4}}$$

$$\text{Using (4) and (3), we get } y = (2)^{(4)^{2/3}} = 2^{\sqrt[3]{16}}$$

5. (2), 6. (3)

We have

$$\begin{aligned} E &= 2 \left(\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{\frac{b}{a}} + \log_b \sqrt[4]{\frac{a}{b}}} \right) \sqrt{\log_a b} \\ &= 2^{\frac{1}{2}} \left(\sqrt{\log_a ab + \log_b ab} - \sqrt{\log_a b/a + \log_b a/b} \right) \sqrt{\log_a b} \\ &= 2^{\frac{1}{2}} \left(\sqrt{2 + \log_a b + \log_b a} - \sqrt{\log_a b + \log_b a - 2} \right) \sqrt{\log_a b} \\ &= 2^{\frac{1}{2}} \left(\sqrt{(\log_a b)^2 + 2\log_a b + 1} - \sqrt{(\log_a b)^2 - 2\log_a b + 1} \right) \\ &= 2^{\frac{1}{2}} \left(\sqrt{(\log_a b + 1)^2} - \sqrt{(\log_a b - 1)^2} \right) \\ &= 2^{\frac{1}{2}} (|\log_a b + 1| - |\log_a b - 1|) \end{aligned}$$

Case I:

$$b \geq a > 1$$

$$\Rightarrow \log_a b \geq \log_a a$$

$$\Rightarrow \log_a b \geq 1$$

$$\Rightarrow E = 2^{\frac{1}{2}(\log_a b + 1 - \log_a b + 1)} = 2$$

Case II:

$$1 < b < a$$

$$\Rightarrow 0 < \log_a b < \log_a a$$

$$\Rightarrow 0 < \log_a b < 1$$

$$\begin{aligned} \Rightarrow E &= 2^{\frac{1}{2}(\log_a b + 1 - 1 + \log_a b)} \\ &= 2^{1/2(2\log_a b)} \\ &= 2^{\log_a b} \end{aligned}$$

Matrix Match Type

$$1. a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r$$

$$a. \text{ We have } \frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4 = \log_\pi 12$$

$$\text{But } \pi^2 < 12 < \pi^3, \text{ we have } 2 < \log_\pi 12 < 3.$$

$$b. 3^a = 4; a = \log_3 4$$

$$\text{Similarly, } b = \log_4 5 \text{ etc.}$$

Hence,

$$\begin{aligned} abcdef &= \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9 \\ &= \log_3 9 = 2 \end{aligned}$$

c. We have to find characteristic of $\log_2 2008$.

$$\text{We know that } \log_2 1024 = 10 \text{ and } \log_2 2048 = 11, \text{ therefore}$$

$$10 < \log_2 2008 < 11$$

Hence, it has characteristic is 10.

$$d. \log_2 (\log_2 (\log_3 x)) = 0$$

$$\Rightarrow \log_2 (\log_3 x) = 1$$

$$\Rightarrow \log_3 x = 2$$

$$\Rightarrow x = 9$$

$$\text{Similarly, we have } \log_3 (\log_2 y) = 1$$

$$\Rightarrow \log_2 y = 3 \text{ or } y = 8$$

$$\text{Therefore, } x - y = 1.$$

$$2. a \rightarrow q; b \rightarrow p; s; c \rightarrow p; d \rightarrow p, r$$

$$a. 2^{\log_{(2/\sqrt{2})} 15} = 2^{\log_{2^{-3/2}} 15} = 2^{23 \log_2 15} = 2^{\log_2 15^{2/3}} = 15^{2/3}$$

$$\begin{aligned} b. & \sqrt[3]{\left(5^{1/\log_7 5} + \frac{1}{\sqrt{(-\log_{10} 0.1)}}\right)} \\ &= \sqrt[3]{\left(5^{\log_5 7} + \frac{1}{\sqrt{(\log_{10} 0.1^{-1})}}\right)} \\ &= \sqrt[3]{\left(7 + \frac{1}{\sqrt{\log_{10} 10}}\right)} = \sqrt[3]{(7+1)} = 2 \end{aligned}$$

$$c. \log_3 5 \cdot \log_{25} 27$$

$$= \frac{\log 5 \log 27}{\log 3 \log 25} = \frac{\log 5}{\log 3} \times \frac{3 \log 3}{2 \log 5} = \frac{3}{2}$$

$$d. (\log_{10} x)^2 = \log 100x = 2 + \log_{10} x$$

$$\text{Putting } \log x = t, \text{ we get } t^2 = 2 + t$$

$$\text{or } t^2 - t - 2 = 0$$

$$\Rightarrow t = 2 \text{ or } t = -1$$

$$\Rightarrow \log_{10} x = 2 \text{ or } \log_{10} x = -1$$

$$\Rightarrow x = 100 \text{ or } x = 1/10$$

Hence, the product of roots is 10.

$$3. (1) a \rightarrow s; b \rightarrow p; c \rightarrow q; d \rightarrow r$$

$$\begin{aligned} a. A &= \log_2 \log_2 \log_4 256 + \log_{\sqrt{2}} (\sqrt{2})^4 \\ &= \log_2 \log_2 \log_4 4^4 + 4 \\ &= \log_2 \log_2 4 + 4 = 1 + 4 = 5 \end{aligned}$$

$$b. \log_3 (5x - 2) - 2 \log_3 \sqrt{3x + 1} = 1 - \log_3 4$$

$$\text{or } \log_3 (5x - 2) - \log_3 (3x + 1) + \log_3 4 = 1$$

$$\text{or } \log_3 \left(\frac{(5x - 2)(4)}{3x + 1} \right) = 1$$

$$\text{or } \frac{(5x - 2)(4)}{3x + 1} = 3$$

$$\text{or } x = 1$$

$$c. 7^{\log_7 (x^2 - 4x + 5)} = (x - 1)$$

$$\text{or } x^2 - 4x + 5 = x - 1$$

$$\text{or } x^2 - 5x + 6 = 0$$

$$\text{or } (x - 2)(x - 3) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 3$$

$$\text{Also we must have } x^2 - 4x + 5 > 0 \text{ and } x - 1 > 0$$

$$\Rightarrow x > 1 \quad (\text{as } x^2 - 4x + 5 > 0 \text{ is true for all real numbers})$$

$$\text{d. } x > 0, \frac{1}{2} \log_2 x - 2 \left(\frac{\log_2 x}{2} \right)^2 + 1 > 0$$

$$\Rightarrow \log_2 x - (\log_2 x)^2 + 2 > 0$$

$$\Rightarrow (\log_2 x)^2 - \log_2 x - 2 < 0$$

$$\text{Let } \log_2 x = t, \text{ we have } t^2 - t - 2 < 0$$

$$\Rightarrow (t-2)(t+1) < 0$$

$$\Rightarrow -1 < t < 2$$

$$\Rightarrow -1 < \log_2 x < 2$$

$$\Rightarrow \frac{1}{2} < x < 4$$

Hence, the number of integers is 3, i.e., $\{1, 2, 3\}$.

Numerical Value Type

$$1. (3) \log_3 c = 3 + \log_3 a$$

$$\text{or } \log_3 \frac{c}{a} = 3 \text{ or } c = 27a$$

$$\log_a b = 2; \log_b c = 2$$

$$\Rightarrow \log_a b \cdot \log_b c = 4$$

$$\Rightarrow \log_a c = 4$$

$$\Rightarrow c = a^4$$

From Eqs. (i) and (ii), we get $a = 3, c = 81$.

From $\log_a b = 2$, we have $b = a^2 = 9$.

Hence, $c/(ab) = 3$.

$$2. (1) \text{ Let } \log_{10} 2 = p \text{ and } \log_{10} 5 = q$$

Hence, $p + q = 1$

$$x = p^3 + 3pq + q^3$$

$$= (p+q)^3 - 3pq(p+q) + 3pq$$

$$= 1 - 3pq + 3pq$$

$$= 1$$

$$3. (6) \text{ Let } \log_4 A = \log_6 B = \log_9 (A+B) = x$$

$$\Rightarrow A = 4^x; B = 6^x \text{ and } A+B = 9^x$$

$$A+B = 9^x \Rightarrow 4^x + 6^x = 9^x$$

$$\Rightarrow 2^{2x} + 2^x \cdot 3^x = 3^{2x}$$

$$\Rightarrow \left(\frac{3}{2}\right)^{2x} - \left(\frac{3}{2}\right)^x - 1 = 0$$

$$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow \frac{B}{A} = \left(\frac{3}{2}\right)^x = \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow 4 \frac{B}{A} = 4 \left(\frac{1+\sqrt{5}}{2} \right)$$

$$\Rightarrow \left[4 \frac{B}{A} \right] = 6$$

$$4. (4) \log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36} \frac{4}{3}$$

$$\text{or } 1 + 2 \log_6 3 + \log_x 16 = 2 \log_2 x - \log_6 \frac{2}{3}$$

$$\text{or } 1 + 2 \log_6 3 + \log_x 16 = 2 \log_2 x - \log_6 2 + \log_6 3$$

$$\text{or } 1 + \log_x 16 = 2 \log_2 x - (\log_6 2 + \log_6 3)$$

$$\text{or } 1 + \log_x 16 = 2 \log_2 x - 1$$

$$\text{or } \frac{4}{\log_2 x} = 2 \log_2 x - 2$$

Let $\log_2 x = t$, we have $(t-2)(t+1) = 0$

$$\Rightarrow t = 2 \text{ or } t = -1$$

$$\Rightarrow x = 4 \text{ or } 1/2$$

$$5. (5) a = \frac{\log_5 175}{\log_5 245} = \frac{2 + \log_5 7}{1 + 2 \log_5 7}$$

$$\text{or } a + 2a \log_5 7 = 2 + \log_5 7$$

$$\text{or } \log_5 7 = \frac{a-2}{1-2a}$$

$$\text{Now } b = \frac{\log_5 875}{\log_5 1715} = \frac{3 + \log_5 7}{1 + 3 \log_5 7}$$

$$\text{or } b + 3b \log_5 7 = 3 + \log_5 7$$

$$\text{or } \log_5 7 = \frac{b-3}{1-3b}$$

From Eqs. (i) and (ii), we get

$$\frac{a-2}{1-2a} = \frac{b-3}{1-3b}$$

$$\Rightarrow \frac{1-ab}{a-b} = 5$$

$$6. (8) (\log_{27} x^3)^2 = \log_{27} x^6$$

$$\text{or } (3 \log_{27} x)^2 = 6 \log_{27} x$$

$$\text{or } 3 \log_{27} x (3 \log_{27} x - 2) = 0$$

$$\Rightarrow x = 1 \text{ or } \log_{27} x = \frac{2}{3}$$

$$\Rightarrow x = (27)^{2/3} = 9$$

$$\text{Difference} = 9 - 1 = 8$$

$$7. (3) \text{ We must have } 12 - 3x > 0$$

$$\text{and } x > 0$$

$$\Rightarrow x \in (0, 4)$$

Therefore, the integral values are 1, 2, 3.

$$\text{For } x = 1; \left(3^{\frac{5}{3} \log_3 9}\right) - (3^{\log_2 1}) = 3^5 - 3^0 > 32$$

Similarly, $x = 2$ satisfies but not $x = 3$

Hence, the required sum = 3.

$$8. (3) \log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 1/6$$

$$= \frac{\log 15}{\log 2} \cdot \frac{\log 2}{-\log 6} \cdot \frac{-\log 6}{\log 3}$$

$$= \log_3 15$$

$$9. (6) \text{ Let } N = 2 \log_x 4 + 3 \log_x 5; \text{ where } x = (2000)^6$$

$$= \log_x 4^2 + \log_x 5^3$$

$$= \log_x 4^2 \cdot 5^3$$

$$= \log_{(2000)^6} (2000)$$

$$= \frac{1}{6}$$

Hence, the reciprocal of given value is 6.

$$10. (5) \sqrt{\log_2 x - 1} - \frac{3}{2} \log_2 x + 2 > 0 \quad (x > 0)$$

$$\Rightarrow \sqrt{\log_2 x - 1} - \frac{3}{2} (\log_2 x - 1) + \frac{1}{2} > 0$$

$$\text{Let } \sqrt{\log_2 x - 1} = t \geq 0, \text{ we have}$$

$$\log_2 x - 1 \geq 0$$

$$\text{Then from Eq. (i), we have } t - \frac{3}{2} t^2 + \frac{1}{2} > 0$$

$$\Rightarrow 3t^2 - 2t - 1 < 0$$

$$\Rightarrow 1/3 < t < 1$$

From Eqs. (ii) and (iii), we have $0 \leq t < 1$.

$$0 \leq \sqrt{\log_2 x - 1} < 1$$

$$0 \leq \log_2 x - 1 < 1$$

$$1 \leq \log_2 x < 2$$

$$2 \leq x < 4$$

Hence, the integral values are 2 and 3, and their sum is 5.

$$11. (2) \log_{1/2} |x-3| > -1$$

$$\Rightarrow 0 < |x-3| < 2$$

$$\Rightarrow -2 < x-3 < 2, x \neq 3$$

$$\Rightarrow 1 < x < 5, x \neq 3$$

$$\therefore x \in \{2, 4\}$$

$$12. (9) \text{ We must have } x > 1$$

$$2 \log_{1/2} (x-1) \leq \frac{1}{3} - \frac{1}{\log_{x^2-x} 8}$$

$$\text{or } \frac{1}{3} - \frac{\log_2 (x^2-x)}{3} + 2 \log_2 (x-1) \geq 0$$

$$\text{or } \log_2 2 - \log_2 (x^2-x) + 6 \log_2 (x-1) \geq 0$$

$$\text{or } \log_2 \frac{2(x-1)^6}{x(x-1)} \geq 0$$

$$\text{or } \frac{2(x-1)^5}{x} \geq 1$$

Putting $x-1 = y$, we have $y > 0$. Thus,

$$\frac{2y^5}{y+1} - 1 \geq 0$$

$$\text{or } \frac{2y^5 - y - 1}{y+1} \geq 0$$

$$\text{or } \frac{2y^5 - 2y + y - 1}{y+1} \geq 0$$

$$\text{or } \frac{2y(y^4 - 1) + y - 1}{y+1} \geq 0$$

$$\text{or } \frac{(y-1)[2y(y+1)(y^2+1)+1]}{y+1} \geq 0$$

$$\text{or } \frac{y-1}{y+1} \geq 0 \text{ or } y \geq 1$$

$$\Rightarrow x \geq 2$$

$$13. (6) 3 + 2\sqrt{2} = (\sqrt{2} + 1)^2 \text{ and } 3 - 2\sqrt{2} = (\sqrt{2} - 1)^2$$

$$\Rightarrow \log_{\left(\sqrt{3+2\sqrt{2}} + \sqrt{3-2\sqrt{2}}\right)} 2^9$$

$$= \frac{1}{\log_2 \left((\sqrt{2} + 1) + (\sqrt{2} - 1) \right)}$$

$$= \frac{1}{\log_2 2^{3/2}}$$

$$= \frac{9}{3/2} = 6$$

$$14. (6) 5^{\log_{1/5} (1/2)} + \log_{\sqrt{2}} \frac{4}{\sqrt{7} + \sqrt{3}} + \log_{1/2} \frac{1}{10 + 2\sqrt{21}}$$

$$= 2 + \log_2 \left(\frac{4}{\sqrt{7} + \sqrt{3}} \right)^2 (\sqrt{7} + \sqrt{3})^2$$

$$= 2 + \log_2 16 = 2 + 4 = 6$$

$$15. (2) N = \frac{\log_5 250}{\log_{50} 5} - \frac{\log_5 10}{\log_{1250} 5}$$

$$= (\log_5 250)(\log_5 50) - (\log_5 10)(\log_5 1250)$$

$$= (3 + \log_5 2)(2 + \log_5 2) - (1 + \log_5 2)(4 + \log_5 2)$$

$$= (\log_5 2)^2 + 5 \log_5 2 + 6 - [(\log_5 2)^2 + 5 \log_5 2 + 4]$$

$$= 2$$

$$16. (1) x + \log_{10} (2^x + 1) = x \log_{10} 5 + \log_{10} 6$$

$$\text{or } x (\log_{10} 10 - \log_{10} 5) + \log_{10} (2^x + 1) = \log_{10} 6$$

$$\text{or } x \log_{10} 2 + \log_{10} (2^x + 1) = \log_{10} 6$$

$$\text{or } \log_{10} 2^x (2^x + 1) = \log_{10} 6$$

$$\text{or } 2^{2x} + 2^x - 6 = 0$$

$$\text{or } (2^x + 3)(2^x - 2) = 0$$

$$\therefore 2^x = 2$$

$$\Rightarrow x = 1$$

$$17. (5) z = 8x^3, z = 5^6 y^6, z = x^{2/3} y^{2/3}$$

$$\Rightarrow x = \frac{z^{1/3}}{2}, y = \frac{z^{1/6}}{5}$$

$$\Rightarrow z = \frac{1}{2^{2/3}} \cdot \frac{z^{2/9}}{5^{2/3}} \cdot \frac{z^{2/18}}{5^{2/3}}$$

$$\Rightarrow z^{2/3} = \frac{1}{(10)^{2/3}} \text{ or } z = \frac{1}{10}$$

$$18. (1) \text{ We have } a = \log_{12} 18 = \frac{\log_2 18}{\log_2 12} = \frac{1 + 2 \log_2 3}{2 + \log_2 3}$$

$$\text{and } b = \log_{24} 54 = \frac{\log_2 54}{\log_2 24} = \frac{1 + 3 \log_2 3}{3 + \log_2 3}$$

Putting $x = \log_2 3$, we have

$$ab + 5(a-b) = \frac{1+2x}{2+x} \cdot \frac{1+3x}{3+x} + 5 \left(\frac{1+2x}{2+x} - \frac{1+3x}{3+x} \right)$$

$$= \frac{6x^2 + 5x + 1 + 5(-x^2 + 1)}{(x+2)(x+3)}$$

$$= \frac{x^2 + 5x + 6}{(x+2)(x+3)} = 1$$

Archives

JEE MAIN

Single Correct Answer Type

$$1. (2) \text{ Let } e^{\sin x} = t$$

$$\Rightarrow t^2 - 4t - 1 = 0$$

$$\Rightarrow t = \frac{4 \pm \sqrt{16+4}}{2}$$

$$\Rightarrow t = e^{\sin x} = 2 \pm \sqrt{5}$$

$$\Rightarrow e^{\sin x} = 2 - \sqrt{5}, e^{\sin x} = 2 + \sqrt{5}$$

$$e^{\sin x} = 2 - \sqrt{5} < 0,$$

$$\Rightarrow \sin x = \ln(2 + \sqrt{5}) > 1$$

So it is rejected, hence there is no solution.

JEE ADVANCED

Single Correct Answer Type

$$1. (3) (2x)^{\ln 2} = (3y)^{\ln 3}$$

$$3^{\ln x} = 2^{\ln y}$$

...(i)

...(ii)

$$\Rightarrow (\log x)(\log 3) = (\log y) \log 2$$

$$\Rightarrow \log y = \frac{(\log x)(\log 3)}{\log 2} \quad \dots(iii)$$

In (i) taking log on both sides, we get

$$(\log 2) \{\log 2 + \log x\} = \log 3 \{\log 3 + \log y\}$$

$$(\log 2)^2 + (\log 2)(\log x) = (\log 3)^2 + \frac{(\log 3)^2(\log x)}{\log 2}$$

[from (iii)]

$$\text{or } (\log 2)^2 - (\log 3)^2 = \frac{(\log 3)^2 - (\log 2)^2}{\log 2} (\log x)$$

$$\text{or } -\log 2 = \log x$$

$$\Rightarrow x = \frac{1}{2}$$

$$\Rightarrow x_0 = \frac{1}{2}$$

Multiple Correct Answers Type

1. (1), (2), (3)

$$3^x = 4^{x-1}$$

$$\Rightarrow \log_2 3^x = (x-1) \log_2 4 = 2(x-1)$$

$$\text{or } x \log_2 3 = 2x - 2$$

$$\text{or } x = \frac{2}{2 - \log_2 3}$$

Rearranging, we get

$$x = \frac{2}{2 - \frac{1}{\log_3 2}} = \frac{2 \log_3 2}{2 \log_3 2 - 1}$$

Rearranging again, we get

$$x = \frac{\log_3 4}{\log_3 4 - 1} = \frac{\frac{1}{\log_4 3}}{\frac{1}{\log_4 3} - 1} = \frac{1}{1 - \log_4 3}$$

Numerical Value Type

1. (4) Let $\sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \dots = y$

$$\text{So, } 4 - \frac{1}{3\sqrt{2}} y = y^2 \quad (y > 0)$$

$$\text{or } y^2 + \frac{1}{3\sqrt{2}} y - 4 = 0$$

$$\text{or } y = \frac{8}{3\sqrt{2}}$$

So, the required value is

$$6 + \log_{3/2} \left(\frac{1}{3\sqrt{2}} \times \frac{8}{3\sqrt{2}} \right) = 6 + \log_3 \frac{4}{9} = 6 - 2 = 4.$$

2. (8) $E = ((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$

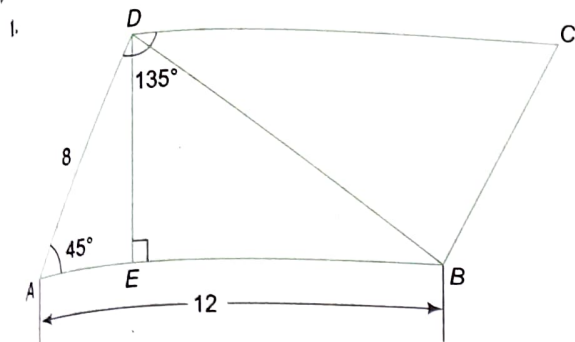
$$\begin{aligned} ((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}} &= ((\log_2 9)^2)^{\log_{(\log_2 9)} 2} \\ &= (\log_2 9)^{2 \log_{(\log_2 9)} 2} \\ &= (\log_2 9)^{\log_{(\log_2 9)} 4} = 4 \end{aligned}$$

$$\begin{aligned} (\sqrt{7})^{\frac{1}{\log_4 7}} &= (7)^{\frac{1}{2} \log_7 4} = 7^{\log_7 2} = 2 \\ \therefore E &= 4 \times 2 = 8 \end{aligned}$$

Chapter 2

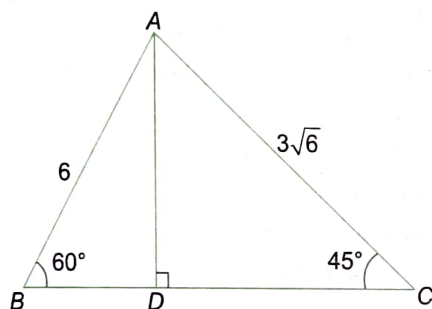
Concept Application Exercises

Exercise 2.1



$$\begin{aligned}\text{Area of the parallelogram} &= 2 \times \text{Area of triangle } ABD \\ &= 2 \times \frac{1}{2} \times AB \times DE \\ &= 12 \times 8 \sin 45^\circ \\ &= \frac{96}{\sqrt{2}} = 48\sqrt{2} \text{ sq. cm}\end{aligned}$$

2. In the figure, $BC = BD + DC$



In triangle ADB ,

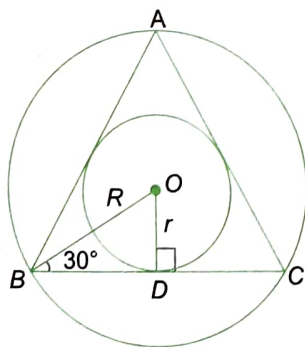
$$BD = AB \cos 60^\circ = 6 \times \frac{1}{2} = 3$$

In triangle ADC ,

$$CD = AC \cos 45^\circ = 3\sqrt{6} \times \frac{1}{\sqrt{2}} = 3\sqrt{3}$$

$$\therefore BC = 3 + 3\sqrt{3}$$

3. In the figure, circumcircle of equilateral triangle ABC has circumference 24π .



Let R be the radius of the circle and O be the centre.

$$\therefore 2\pi R = 24\pi$$

$$\Rightarrow R = 12$$

Circle inscribed in triangle touches BC at D , which is midpoint of BC .

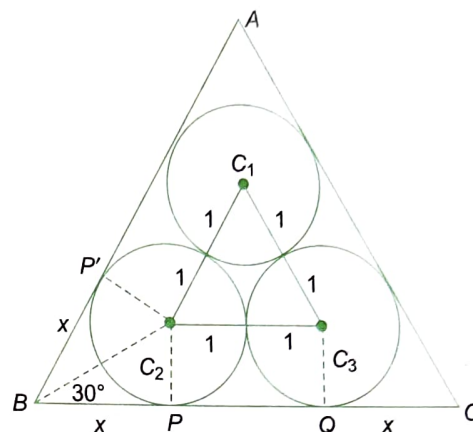
In triangle ODB ,

$$\sin 30^\circ = \frac{OD}{OB} = \frac{r}{R} \quad (r \text{ is the radius of incircle})$$

$$\therefore r = \frac{12}{2} = 6$$

$$\therefore \text{Area of inscribed circle} = \pi r^2 = 36\pi \text{ sq. unit}$$

4. The given arrangement of coins is as shown in the given figure.



To find the area of the triangle, we need to find its side.

For circle with centre C_2 , BP and BP' are two tangents from B to circle,

Therefore, BC_2 must be the angle bisector of $\angle B$.

$$\angle B = 60^\circ \quad (\because \triangle ABC \text{ is equilateral})$$

$$\therefore \angle C_2BP = 30^\circ$$

$$\therefore \tan 30^\circ = \frac{1}{x}$$

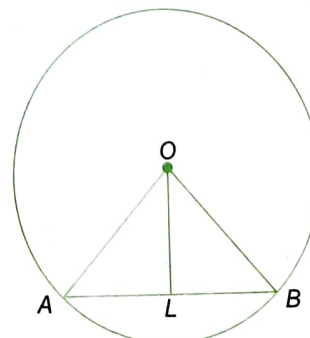
$$\Rightarrow x = \sqrt{3}$$

$$\begin{aligned}\therefore BC &= BP + PQ + QC \\ &= x + 2 + x = 2 + 2\sqrt{3}\end{aligned}$$

Area of an equilateral triangle is $\frac{\sqrt{3}}{4} a^2$, where a is side length.

$$\begin{aligned}\therefore \text{Area of } \triangle ABC &= \frac{\sqrt{3}}{4} (2 + 2\sqrt{3})^2 \\ &= \sqrt{3} (1 + 3 + 2\sqrt{3}) \\ &= (4\sqrt{3} + 6) \text{ sq. unit}\end{aligned}$$

5. Let $AB = 2$ units be one of the sides of the polygon.



Then $\angle AOB = 360^\circ/9 = 40^\circ$, where O is the centre of circle.

If $OL \perp AB$, then $AL = 1$ and $\angle AOL = 20^\circ$

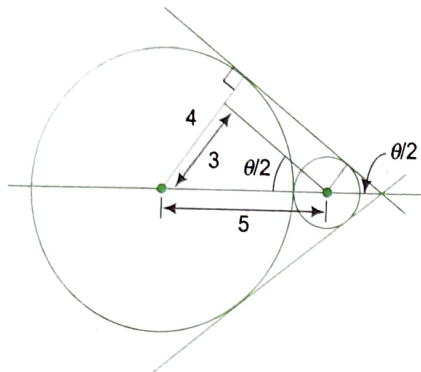
$$\therefore \text{Radius of the circle} = OA = AL \operatorname{cosec} 20^\circ = \operatorname{cosec} 20^\circ$$

6. From figure, we have

$$\sin \frac{\theta}{2} = \frac{3}{5}$$

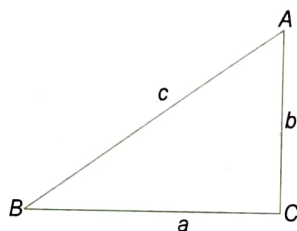
$$\cos \frac{\theta}{2} = \frac{4}{5}$$

$$\therefore \sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \frac{7}{5}$$



7. Draw $\triangle ABC$ with $\angle C = 90^\circ$. We have

$$\tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$$



$$8. \cos^2 \alpha - \sin^2 \alpha = \tan^2 \beta$$

$$\Rightarrow \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos^2 \alpha + \sin^2 \alpha} = \frac{\sin^2 \beta}{\cos^2 \beta}$$

$$\Rightarrow \frac{\cos^2 \alpha - \sin^2 \alpha + \cos^2 \alpha + \sin^2 \alpha}{\cos^2 \alpha - \sin^2 \alpha - \cos^2 \alpha - \sin^2 \alpha} = \frac{\sin^2 \beta + \cos^2 \beta}{\sin^2 \beta - \cos^2 \beta}$$

$$\Rightarrow \frac{1}{\tan^2 \alpha} = \frac{1}{\cos^2 \beta - \sin^2 \beta}$$

$$\therefore \tan^2 \alpha = \cos^2 \beta - \sin^2 \beta$$

Exercise 2.2

$$\begin{aligned} 1. \frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} &= \frac{(\sin^2 x) - (\cos x - 1)^2}{(\sin x + \cos x - 1)^2} \\ &= \frac{2 \cos x - 2 \cos^2 x}{2 + 2 \sin x \cos x - 2 \cos x - 2 \sin x} \\ &= \frac{2 \cos x(1 - \cos x)}{2(1 - \sin x)(1 - \cos x)} \\ &= \frac{\cos x}{1 - \sin x} = \frac{(1 + \sin x)}{\cos x} = \sec x + \tan x \end{aligned}$$

2. Divide by $\cos^4 \alpha$

$$\Rightarrow 15 \tan^4 \alpha + 10 = 6 \sec^4 \alpha$$

$$\Rightarrow 15 \tan^4 \alpha + 10 = 6(1 + \tan^2 \alpha)^2$$

$$\Rightarrow (3 \tan^2 \alpha - 2)^2 = 0$$

$$\therefore \tan^2 \alpha = 2/3$$

$$\begin{aligned} \therefore 8 \operatorname{cosec}^2 \alpha + 27 \sec^2 \alpha &= \sec^2 \alpha (8 \cot^2 \alpha + 27) \\ &= \frac{5}{3} \left(8 \times \frac{3}{2} + 27 \right) = 65 \end{aligned}$$

$$3. \sec \theta + \tan \theta = p$$

$$\Rightarrow \sec \theta - \tan \theta = \frac{1}{p}$$

Subtracting second from first, we get

$$2 \tan \theta = p - \frac{1}{p} \quad \text{or} \quad \tan \theta = \frac{p^2 - 1}{2p}$$

$$4. \text{ Multiplying both sides by } (1 - \sin A)(1 - \sin B)(1 - \sin C), \text{ we get}$$

$$(1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 C)$$

$$= (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$$

$$\Rightarrow (1 - \sin A)(1 - \sin B)(1 - \sin C)$$

$$= \pm \cos A \cos B \cos C$$

$$\text{Similarly, } (1 + \sin A)(1 + \sin B)(1 + \sin C)$$

$$= \pm \cos A \cos B \cos C$$

$$5. \text{ Given } (\sec \theta + \tan \theta)(\sec \phi + \tan \phi)(\sec \psi + \tan \psi)$$

$$= \tan \theta \tan \phi \tan \psi$$

$$\text{Since } (\sec \theta - \tan \theta) = \frac{1}{(\sec \theta + \tan \theta)}, \text{ we have}$$

$$(\sec \theta - \tan \theta)(\sec \phi - \tan \phi)(\sec \psi - \tan \psi)$$

$$= \frac{1}{(\sec \theta + \tan \theta)(\sec \phi + \tan \phi)(\sec \psi + \tan \psi)}$$

$$= \frac{1}{\tan \theta \tan \phi \tan \psi} = \cot \theta \cot \phi \cot \psi$$

$$6. \text{ Squaring and adding, we have } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

7. The given equations can be rewritten as

$$a \cos \theta + b \sin \theta = 1 \text{ and } -a \sin \theta + b \cos \theta = 3$$

$$\text{Squaring and adding, we have } a^2 + b^2 = 10$$

$$8. a \sin^2 x + b(1 - \sin^2 x) = c$$

$$\therefore (a - b) \sin^2 x = c - b$$

$$\text{Also, } a(1 - \cos^2 x) + b \cos^2 x = c$$

$$\therefore (b - a) \cos^2 x = c - a$$

$$\therefore \tan^2 x = \frac{c - b}{a - c}$$

$$\text{Similarly, } \tan^2 y = \frac{d - a}{b - d}$$

$$\therefore \frac{a^2}{b^2} = \frac{\tan^2 y}{\tan^2 x} = \frac{d - a}{b - d} \times \frac{c - a}{b - c}$$

Exercise 2.3

1. Let the post be at point P and PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72^\circ$ and arc $AB = 88$ m.

Let r be the length of the rope, i.e., $PA = r$ meters.

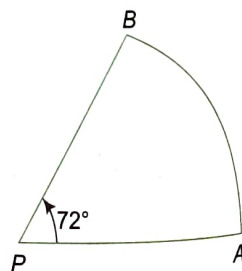
$$\begin{aligned} \text{Now, } \theta = 72^\circ &= \left(72 \times \frac{\pi}{180} \right)^R \\ &= \left(\frac{2\pi}{5} \right)^R \end{aligned}$$

$$\text{and } s = 88 \text{ m}$$

$$\therefore \theta = \frac{\text{Arc}}{\text{Radius}}$$

$$\text{or } \frac{2\pi}{5} = \frac{88}{r}$$

$$\text{or } r = 88 \times \frac{5}{2\pi} = 70 \text{ m}$$



Suppose the coin is kept at a distance r from the eye to hide the moon completely.
Let E be the eye of the observer and AB be the diameter of the coin.

Then, arc AB = Diameter $AB = 2.2$ cm.

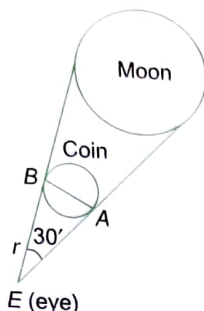
$$\text{Now, } \theta = 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2} \times \frac{\pi}{180}\right)^R = \left(\frac{\pi}{360}\right)^R$$

$$\therefore \theta = \frac{\text{Arc}}{\text{Radius}}$$

$$\text{or } \frac{\pi}{360} = \frac{2.2}{r}$$

$$\text{or } r = \frac{2.2 \times 360}{\pi} \text{ cm}$$

$$= \frac{2.2 \times 360 \times 7}{22} = 252 \text{ cm}$$



3. The angle traced by the hour hand in 12 h = 360°

Therefore, the angle traced by the hour hand in 3 h 30 min, i.e.,

$$\frac{7}{2} \text{ h} = \left(\frac{360}{12} \times \frac{7}{2}\right)^\circ = 105^\circ$$

The angle traced by the minute hand in 60 min = 360°

The angle traced by the minute hand in 30 min

$$= \left(\frac{360}{60} \times 30\right)^\circ = 180^\circ$$

Hence, required angle between the two hands

$$= 180^\circ - 105^\circ = 75^\circ = \left(75 \times \frac{\pi}{180}\right) = \frac{5\pi}{12} \text{ rad}$$

4. From the question, $\frac{\sqrt{3}}{4} \times 16 = 2 \times \frac{1}{2} r^2 \times \frac{\pi}{3}$

$$\pi r^2 = 12\sqrt{3} \quad \text{or} \quad r = \sqrt{\frac{12\sqrt{3}}{\pi}}$$

Exercise 2.4

1. Here $x = -3$, $y = -4$,

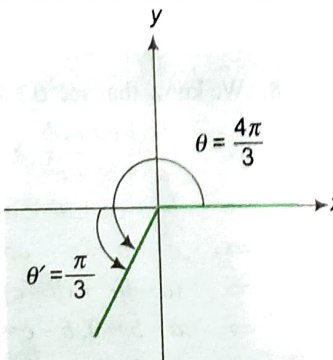
Terminal angle θ lies in the third quadrant.

$$\text{Now, } r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

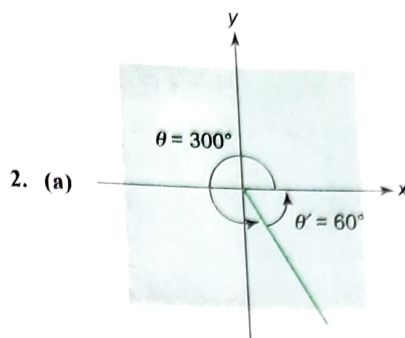
$$\therefore \sin \theta = \frac{y}{r} = -\frac{4}{5}$$

$$\cos \theta = \frac{x}{r} = -\frac{3}{5}$$

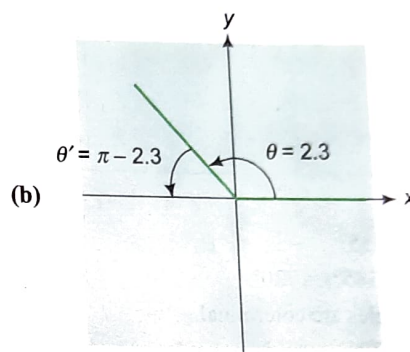
3.

θ	Quadrant	Reference angle (θ')	Required value
(a) $\frac{4\pi}{3}$		$\frac{\pi}{3}$	$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$

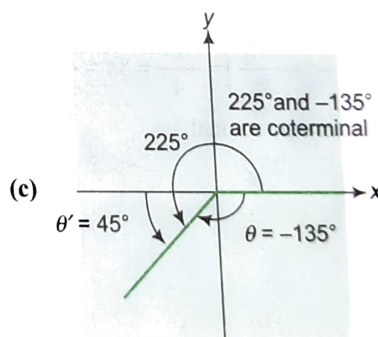
$$\tan \theta = \frac{y}{x} = \frac{4}{3}$$



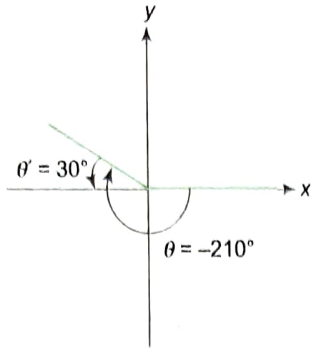
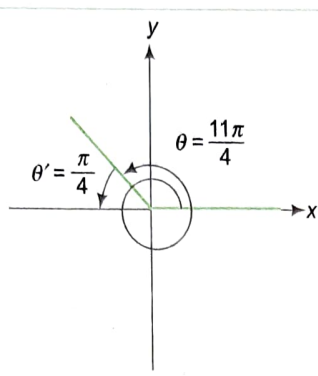
Because 300° lies in Quadrant IV, the acute angle it makes with the x -axis is $\theta' = 360^\circ - 300^\circ = 60^\circ$.



Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is $\theta' = \pi - 2.3 \approx 0.8416$.



First, determine that -135° is coterminal with 225° , which lies in Quadrant III. So, the reference angle is $\theta' = 225^\circ - 180^\circ = 45^\circ$.

(b) -210°		30°	$\tan(-210^\circ) = -\tan 30^\circ$ $= -\frac{1}{\sqrt{3}}$
(c) $\frac{11\pi}{4}$		$\frac{\pi}{4}$	$\operatorname{cosec} \frac{11\pi}{4} = \operatorname{cosec} \frac{\pi}{4}$ $= \sqrt{2}$

4. (a) $\alpha = -185^\circ, \beta = 535^\circ$
 $\beta - \alpha = 535^\circ - (-185^\circ) = 720^\circ$
Hence, given angles are coterminal.

- (b) $\alpha = 1000^\circ, \beta = 270^\circ$
 $\alpha - \beta = 1000^\circ - 270^\circ = 730^\circ$
Hence, given angles are not coterminal.

- (c) $\alpha = \frac{15\pi}{4}, \beta = -\frac{17\pi}{4}$

$$\text{Here } \alpha - \beta = \frac{15\pi}{4} - \left(-\frac{17\pi}{4}\right) = \frac{32\pi}{4} = 8\pi = 4(2\pi)$$

Hence, given angles are coterminal.

Exercise 2.5

1. $-1 \leq \sin x \leq 1$
 $\Rightarrow 2 \leq 3 + \sin x \leq 4$
 $\Rightarrow \frac{1}{4} \leq \frac{1}{\sin x + 3} \leq \frac{1}{2}$
 $\Rightarrow 2 \leq \frac{8}{\sin x + 3} \leq 4$

2. $f(x) = \sin(\cos x)$

For $\forall x \in R, \theta = \cos x \in [-1, 1]$

Since $\sin \theta$ is increasing from $-\pi/2$ to $\pi/2$, the maximum value occurs at $\theta = 1$ and the minimum value occurs at $\theta = -1$. Hence, range is $[-\sin 1, \sin 1]$.

3. $f(\theta) = 12 \sin \theta - 9 \sin^2 \theta$

$$= -(9 \sin^2 \theta - 12 \sin \theta)$$

$$= -(9 \sin^2 \theta - 12 \sin \theta + 4 - 4)$$

$$= -((3 \sin \theta - 2)^2 - 4) = 4 - (3 \sin \theta - 2)^2$$

Minimum value of $f(\theta)$ occurs when $(3 \sin \theta - 2)^2$ is maximum, which is $(-3 - 2)^2 = 25$.

Maximum value of $f(\theta)$ occurs when $(3 \sin \theta - 2)^2$ is minimum, which is 0.

Hence, range of $f(\theta)$ is $[4 - 25, 4] \equiv [-21, 4]$.

4. $9 \tan^2 \theta + 4 \cot^2 \theta = (3 \tan \theta - 2 \cot \theta)^2 + 12$

Hence, the minimum value is 12.

5. (a) $\sin \theta = \frac{n+1}{n} = 1 + \frac{1}{n} > 1$, which is not true.

- (b) $\sin \theta = \frac{n^2+1}{n+1} > 1$, which is not true.

- (c) $\sec \theta = \frac{n+2}{n-1} > 1$, which is true.

- (d) $\sec \theta = \frac{n}{\sqrt{n^2+1}} < 1$, which is not true.

6. $\sin^2 \theta_1 + \sin^2 \theta_2 + \dots + \sin^2 \theta_n = 0$

$$\Rightarrow \sin \theta_1 = \sin \theta_2 = \dots = \sin \theta_n = 0$$

$$\Rightarrow \cos \theta_1, \cos \theta_2, \dots, \cos \theta_n = \pm 1$$

Therefore, the minimum value of $\cos \theta_1 + \cos \theta_2 + \dots + \cos \theta_n$
 $= (-1) + (-1) + (-1) + \dots n \text{ times} = -n$

7. $\sin^2 \theta = x^2 - 3x + 3$

Now, $\sin^2 \theta \in [0, 1]$

$$\Rightarrow x^2 - 3x + 3 \in [0, 1]$$

$$\Rightarrow 0 \leq x^2 - 3x + 3 \leq 1$$

Now, $x^2 - 3x + 3 \geq 0$ is always true. Thus,

$$x^2 - 3x + 3 \leq 1$$

$$\text{or } x^2 - 3x + 2 \leq 0$$

$$\text{or } (x-1)(x-2) \leq 0$$

$$\text{or } x \in [1, 2]$$

8. We know that $\sec^2 \theta \geq 1$

$$\therefore \frac{bc + ca + ab}{a^2 + b^2 + c^2} \geq 1$$

$$\Rightarrow bc + ca + ab \geq a^2 + b^2 + c^2$$

$$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca \leq 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \leq 0$$

$$\Rightarrow a-b=0, b-c=0 \text{ and } c-a=0$$

$$\Rightarrow a=b=c$$

$$\begin{aligned}
 9. f(x) &= \sqrt{4 - \sqrt{1 + \tan^2 x}} \\
 &= \sqrt{4 - \sqrt{\sec^2 x}} \\
 &= \sqrt{4 - |\sec x|} \\
 |\sec x| &\geq 1 \\
 -|\sec x| &\leq -1 \\
 4 - |\sec x| &\leq 3 \\
 \Rightarrow \sqrt{4 - \sqrt{1 + \tan^2 x}} &\in [0, \sqrt{3}]
 \end{aligned}$$

$$\begin{aligned}
 10. 0 &\leq |\cos x| \leq 1 \\
 \Rightarrow 0 &\leq 2|\cos x| \leq 2 \\
 \Rightarrow -3 &\leq 2|\cos x| - 3 \leq -1 \\
 \Rightarrow \frac{1}{2|\cos x| - 3} &\in [-1, -1/3]
 \end{aligned}$$

$$\begin{aligned}
 11. f(x) &= \cos^4 x + \sin^2 x - 1 \\
 &= \cos^4 x + 1 - \cos^2 x - 1 \\
 &= \cos^4 x - \cos^2 x \\
 &= (\cos^2 x - 1/2)^2 - 1/4
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } 0 &\leq \cos^2 x \leq 1 \\
 \Rightarrow -1/2 &\leq \cos^2 x - 1/2 \leq 1/2 \\
 \Rightarrow -1/4 &\leq (\cos^2 x - 1/2)^2 - 1/4 \leq 0
 \end{aligned}$$

12. Since $0 \leq 1 + \sin x \leq 2$ and $0 \leq 1 + \cos x \leq 2$,
Minimum value of $f(x)$ is 0, when any one of $(1 + \sin x)$ or $(1 + \cos x)$ is zero.

$$\begin{aligned}
 13. (\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 \\
 &= \sin^2 \theta + \operatorname{cosec}^2 \theta + 2 + \cos^2 \theta + \sec^2 \theta + 2 \\
 &= (\sin^2 \theta + \cos^2 \theta) + (\operatorname{cosec}^2 \theta + \sec^2 \theta) + 4 \\
 &= 5 + 1 + \tan^2 \theta + 1 + \cot^2 \theta \\
 &= 7 + (\tan \theta - \cot \theta)^2 + 2 \\
 &= 9 + (\tan \theta - \cot \theta)^2 \geq 9
 \end{aligned}$$

$$\begin{aligned}
 14. f(x) &= (1 + \cot^2 x) + 25(1 + \tan^2 x) \\
 &= 26 + \cot^2 x + 25 \tan^2 x \\
 &= 26 + 10 + (\cot^2 x + 25 \tan^2 x - 10) \\
 &= 36 + (\cot x - 5 \tan x)^2 \geq 36
 \end{aligned}$$

$$15. \cos^2 x + \cos x = a + 2$$

$$\Rightarrow \left(\cos x + \frac{1}{2} \right)^2 = a + \frac{9}{4}$$

$$\text{Now } -\frac{1}{2} \leq \cos x + \frac{1}{2} \leq \frac{3}{2}$$

$$\therefore \left(\cos x + \frac{1}{2} \right)^2 \in \left[0, \frac{9}{4} \right]$$

$$\text{So, } 0 \leq a + \frac{9}{4} \leq \frac{9}{4}$$

$$\therefore -\frac{9}{4} \leq a \leq 0$$

$$16. \text{ We have } (a+1)^2 + \operatorname{cosec}^2 \left(\frac{\pi}{2}(a+x) \right) - 1 = 0$$

$$\text{or } (a+1)^2 + \cot^2 \left(\frac{\pi}{2}(a+x) \right) = 0$$

$$\Rightarrow a = -1 \text{ and } \cot^2 \left(\frac{-\pi}{2} + \frac{\pi x}{2} \right) = 0$$

$$\Rightarrow \tan^2 \left(\frac{\pi x}{2} \right) = 0 \text{ or } \tan \left(\frac{\pi x}{2} \right) = 0$$

$$\Rightarrow \frac{\pi x}{2} = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow x = 2n, n \in \mathbb{Z}$$

Exercise 2.6

$$\begin{aligned}
 1. (a) \tan 720^\circ - \cos 270^\circ - \sin 150^\circ \cos 120^\circ \\
 = 0 - 0 - (\sin 30^\circ)(-\cos 60^\circ) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 (b) \sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 150^\circ \\
 = \sin(720^\circ + 60^\circ) \sin(360^\circ + 120^\circ) \\
 + \cos(180^\circ - 60^\circ) \sin(180^\circ - 30^\circ) \\
 = \sin(60^\circ) \sin(120^\circ) - \cos(60^\circ) \sin(30^\circ) \\
 = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 2. \sec 610^\circ \operatorname{cosec} 160^\circ - \cot 380^\circ \tan 470^\circ \\
 = \sec(720^\circ - 110^\circ) \operatorname{cosec}(180^\circ - 20^\circ) \\
 - \cot(360^\circ + 20^\circ) \tan(360^\circ + 110^\circ) \\
 = \frac{1}{\cos 110^\circ} \cdot \frac{1}{\sin 20^\circ} - \cot 20^\circ \tan 110^\circ \\
 = \frac{-1}{\sin^2 20^\circ} + \cot^2 20^\circ \\
 = -[\operatorname{cosec}^2 20^\circ - \cot^2 20^\circ] \\
 = -1
 \end{aligned}$$

$$\begin{aligned}
 3. \cos \alpha \times \cos 2\alpha \times \cos 3\alpha \times \cos 4\alpha \times \cos 5\alpha \times \cos 6\alpha \\
 = \cos \frac{\pi}{3} \times \cos \frac{2\pi}{3} \times \cos \pi \times \cos \frac{4\pi}{3} \times \cos \frac{5\pi}{3} \times \cos \frac{6\pi}{3} \\
 = \frac{1}{2} \times \left(-\frac{1}{2} \right) (-1) \left(-\frac{1}{2} \right) \left(\frac{1}{2} \right) 1 \\
 = -\frac{1}{16}
 \end{aligned}$$

$$\begin{aligned}
 4. \tan \frac{\pi}{20} \tan \frac{3\pi}{20} \tan \frac{5\pi}{20} \tan \frac{7\pi}{20} \tan \frac{9\pi}{20} \\
 = \tan \frac{\pi}{20} \tan \frac{3\pi}{20} \tan \frac{\pi}{4} \tan \left(\frac{\pi}{2} - \frac{3\pi}{20} \right) \tan \left(\frac{\pi}{2} - \frac{\pi}{20} \right) \\
 = \tan \frac{\pi}{20} \tan \frac{3\pi}{20} (1) \cot \left(\frac{3\pi}{20} \right) \cot \left(\frac{\pi}{20} \right) \\
 = 1
 \end{aligned}$$

$$\begin{aligned}
 5. \frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} &= \frac{\cot(90^\circ - 36^\circ)}{\tan 36^\circ} + \frac{\cot(90^\circ - 70^\circ)}{\cot 70^\circ} \\
 &= \frac{\tan 36^\circ}{\tan 36^\circ} + \frac{\cot 70^\circ}{\cot 70^\circ} = 2
 \end{aligned}$$

$$\begin{aligned}
 6. \sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} \\
 = \sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \left(\frac{\pi}{2} - \frac{\pi}{9} \right) + \sin^2 \left(\frac{\pi}{2} - \frac{\pi}{18} \right) \\
 = \sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \cos^2 \frac{\pi}{9} + \cos^2 \frac{\pi}{18} \\
 = \left(\sin^2 \frac{\pi}{18} + \cos^2 \frac{\pi}{18} \right) + \left(\sin^2 \frac{\pi}{9} + \cos^2 \frac{\pi}{9} \right) = 2
 \end{aligned}$$

$$7. \sec\left(\frac{3\pi}{2} - \theta\right) \sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right) \tan\left(\theta - \frac{3\pi}{2}\right)$$

$$= -\operatorname{cosec} \theta \operatorname{cosec} \theta + \cot \theta \cot \theta$$

$$= -(\operatorname{cosec}^2 \theta - \cot^2 \theta) = -1$$

$$8. 2n\theta = \pi/2$$

$$\therefore \theta, (2n-1)\theta = (\pi/2) - \theta$$

$$\text{or } 2\theta, (2n-2)\theta = (\pi/2) - 2\theta, \dots$$

They form complementary angles A and B so that $\tan A \tan B = \tan A \cot A = 1$ for each pair.

$$\text{Also, } \tan n\theta = \tan \frac{\pi}{4} = 1$$

Hence, the value of product is 1

$$9. \text{ In quadrilateral } ABCD, A + B + C + D = 2\pi.$$

$$(a) \sin(A+B) + \sin(C+D) = \sin(A+B) + \sin(2\pi - (A+B))$$

$$= \sin(A+B) - \sin(A+B) = 0$$

$$(b) \cos(A+B) = \cos(C+D)$$

$$= \cos(2\pi - (C+D)) = \cos(C+D)$$

Exercises

Single Correct Answer Type

$$1. (3) 5 \tan \theta = 4 \text{ or } \tan \theta = \frac{4}{5}$$

$$\text{Now, } \frac{5 \sin \theta - 3 \cos \theta}{5 \sin \theta + 2 \cos \theta} = \frac{5 \frac{\sin \theta}{\cos \theta} - 3}{5 \frac{\sin \theta}{\cos \theta} + 2}$$

$$= \frac{5 \tan \theta - 3}{5 \tan \theta + 2}$$

$$= \frac{5 \times \frac{4}{5} - 3}{5 \times \frac{4}{5} + 2} = \frac{1}{6}$$

$$2. (2) \tan \theta = \frac{-4}{3}$$

$$\Rightarrow \theta \in 2\text{nd quadrant or } 4\text{th quadrant}$$

$$\Rightarrow \sin \theta = \pm 4/5$$

$$\text{If } \theta \in 2\text{nd quadrant, } \sin \theta = 4/5$$

$$\text{If } \theta \in 4\text{th quadrant, } \sin \theta = -4/5$$

$$3. (1) \sin x + \operatorname{cosec} x = 2$$

$$\text{or } (\sin x - 1)^2 = 0$$

$$\text{or } \sin x = 1$$

$$\Rightarrow \sin^n x + \operatorname{cosec}^n x = 1 + 1 = 2$$

$$4. (4) \text{ From the given relations,}$$

$$m + n = 2 \tan \theta, m - n = 2 \sin \theta.$$

$$\text{Thus, } m^2 - n^2 = 4 \tan \theta \sin \theta$$

$$\text{Also } \sqrt{mn} = \sqrt{\tan^2 \theta - \sin^2 \theta} = \sin \theta \tan \theta$$

$$\text{From Eqs. (i) and (ii), we get } m^2 - n^2 = 4\sqrt{mn}.$$

$$5. (3) \operatorname{cosec} \theta - \cot \theta = q$$

$$\therefore \operatorname{cosec} \theta + \cot \theta = \frac{1}{q}$$

$$\therefore \operatorname{cosec} \theta = \frac{1}{2} \left(q + \frac{1}{q} \right)$$

(on addition)

$$6. (2) bc + \frac{1}{ck} + \frac{ak}{1+bk} = \frac{\cos x \tan x}{k^2} + \frac{1}{\tan x} + \frac{\sin x}{1 + \cos x}$$

$$= \frac{\sin x}{k^2} + \frac{\cos x(1 + \cos x) + \sin^2 x}{\sin x(1 + \cos x)}$$

$$= \frac{a}{k} + \frac{\cos x(1 + \cos x) + (1 - \cos^2 x)}{\sin x(1 + \cos x)}$$

$$= \frac{a}{k} + \frac{1}{\sin x}$$

$$= \frac{a}{k} + \frac{1}{ak}$$

$$7. (3) \sec^4 \theta + \sec^2 \theta = 10 + \tan^4 \theta + \tan^2 \theta$$

$$\Rightarrow \sec^4 \theta - \tan^4 \theta + \sec^2 \theta - \tan^2 \theta = 10$$

$$\Rightarrow \sec^2 \theta + \tan^2 \theta + 1 = 10$$

$$\Rightarrow 2 \sec^2 \theta = 10$$

$$\Rightarrow \cos^2 \theta = \frac{1}{5}$$

$$\Rightarrow \sin^2 \theta = \frac{4}{5}$$

$$8. (3) \text{ Multiplying the numerator and denominator of } x \text{ by } 1 - \cos \theta + \sin \theta, \text{ we get}$$

$$x = \frac{2 \sin \theta (1 - \cos \theta + \sin \theta)}{(1 + \sin \theta)^2 - \cos^2 \theta}$$

$$= \frac{2 \sin \theta (1 - \cos \theta + \sin \theta)}{(1 + \sin \theta)^2 - (1 - \sin^2 \theta)}$$

$$= \frac{2 \sin \theta \cdot (1 - \cos \theta + \sin \theta)}{(1 + \sin \theta)(2 \sin \theta)}$$

$$= \frac{1 - \cos \theta + \sin \theta}{1 + \sin \theta}$$

$$9. (2) \sec \alpha + \operatorname{cosec} \alpha = p, \sec \alpha \operatorname{cosec} \alpha = q$$

$$\text{or } \frac{\sin \alpha + \cos \alpha}{\sin \alpha \cos \alpha} = p \text{ and } \frac{1}{\sin \alpha \cos \alpha} = q$$

$$\text{or } \frac{1 + 2 \sin \alpha \cos \alpha}{\sin^2 \alpha \cos^2 \alpha} = p^2$$

$$\text{or } \frac{1 + \frac{2}{q}}{\frac{1}{q^2}} = p^2$$

$$\text{or } q^2 \left(1 + \frac{2}{q} \right) = p^2 \text{ or } q(q+2) = p^2$$

$$10. (4) \sec^2 \theta + \operatorname{cosec}^2 \theta = \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}$$

$$= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta \sin^2 \theta} = \sec^2 \theta \operatorname{cosec}^2 \theta$$

$$\text{Now, } \sec^2 \theta + \operatorname{cosec}^2 \theta = 2 + \tan^2 \theta + \cot^2 \theta$$

$$= 4 + (\tan \theta - \cot \theta)^2$$

$$\geq 4$$

Hence, all equations can have roots $\sec^2 \theta$ and $\operatorname{cosec}^2 \theta$.

14. (1) We have $\sin x + \sin^2 x = 1$
 or $\sin x = 1 - \sin^2 x$ or $\sin x = \cos^2 x$

$$\begin{aligned} \text{Now } \cos^{12} x + 3 \cos^{10} x + 3 \cos^8 x + \cos^6 x - 2 \\ = \sin^6 x + 3 \sin^5 x + 3 \sin^4 x + \sin^3 x - 2 \\ = (\sin^2 x)^3 + 3(\sin^2 x)^2 \sin x + 3(\sin^2 x) \end{aligned}$$

$$= (\sin^2 x + \sin x)^3 - 2 = (1)^3 - 2 = -1$$

$$\begin{aligned} 14. (3) 3(\sin \theta - \cos \theta)^4 + 6(\sin \theta + \cos \theta)^2 + 4(\sin^6 \theta + \cos^6 \theta) \\ = 3[\sin^4 \theta + \cos^4 \theta - 4 \sin^3 \theta \cos \theta \\ + 6 \sin^2 \cos^2 \theta - 4 \sin \theta \cos^3 \theta] + 6[1 + 2 \sin \theta \cos \theta] \\ + 4[\sin^4 \theta + \cos^4 \theta - \sin^2 \theta \cos^2 \theta] \\ = 7[\sin^4 \theta + \cos^4 \theta] + 14 \sin^2 \theta \cos^2 \theta \\ - 12 \sin \theta \cos \theta + 6 + 12 \sin \theta \cos \theta \\ = 7(\sin^2 \theta + \cos^2 \theta)^2 + 6 = 13 \end{aligned}$$

$$15. (3) \sec x = 1 - \sec^2 x$$

$$\Rightarrow \sec x = -\tan^2 x$$

Squaring both side, we get

$$\sec^2 x = \tan^4 x$$

$$\Rightarrow 1 + \tan^2 x = \tan^4 x$$

Squaring both side, we get

$$\tan^8 x = \tan^4 x + 2 \tan^2 x + 1$$

$$\Rightarrow \tan^8 x - \tan^4 x - 2 \tan^2 x = 1$$

$$\Rightarrow \tan^8 x - \tan^4 x - 2 \tan^2 x + 1 = 2$$

$$\begin{aligned} 14. (2) (1 + \tan \alpha \tan \beta)^2 + (\tan \alpha - \tan \beta)^2 \\ = 1 + 2 \tan \alpha \tan \beta + \tan^2 \alpha \tan^2 \beta + \tan^2 \alpha + \\ \tan^2 \beta - 2 \tan \alpha \tan \beta \\ = 1 + \tan^2 \alpha \tan^2 \beta + \tan^2 \alpha + \tan^2 \beta \\ = (1 + \tan^2 \alpha)(1 + \tan^2 \beta) \\ = \sec^2 \alpha \sec^2 \beta \end{aligned}$$

15. (3) Let O be the center of the circle.

$$\angle A_0 O A_1 = \frac{360^\circ}{6} = 60^\circ$$

Thus, $A_0 O A_1$ is an equilateral triangle. We get

$$A_0 A_1 = 1$$

(radius of circle = 1)

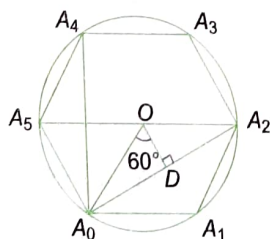
$$\text{Also } A_0 A_2 = A_0 A_4$$

$$= 2A_0 D$$

$$= 2(OA_0 \sin 60^\circ)$$

$$= 2(1) \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\therefore (A_0 A_1)(A_0 A_2)(A_0 A_4) \\ = (1)(\sqrt{3})(\sqrt{3}) = 3$$

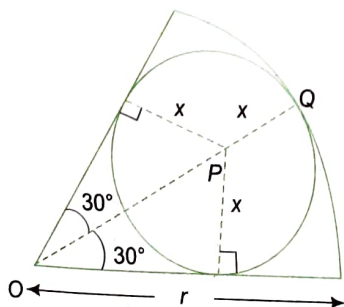


16. (2) Let the radius of smaller circle be x . Here,

$$OP = x \operatorname{cosec} 30^\circ$$

$$\text{and } OQ = r = x + x \operatorname{cosec} 30^\circ$$

$$\Rightarrow x = \frac{r}{3}$$

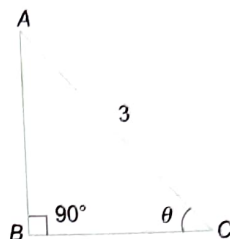


17. (4) Perimeter = $7 = 3 \sin \theta + 3 \cos \theta + 3$, where $45^\circ < \theta < 90^\circ$

$$\sin \theta + \cos \theta = \frac{4}{3} \quad \dots(i)$$

Since $(\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2$, we have

$$(\sin \theta - \cos \theta)^2 = 2 - \left(\frac{4}{3}\right)^2$$



$$\Rightarrow \sin \theta - \cos \theta = \frac{\sqrt{2}}{3} \quad \dots(ii)$$

$$\text{From (i) and (ii), } \cos \theta = \frac{4 - \sqrt{2}}{6}$$

18. (2) Area of rhombus is given by

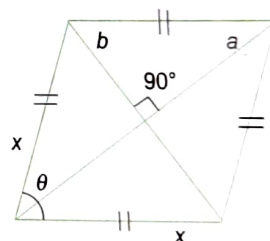
$$2 \times \frac{1}{2} x^2 \sin \theta = \frac{ab}{2} \quad (\text{Equating area of a rhombus})$$

$$\text{Given } x^2 = ab$$

(given)

$$\Rightarrow \sin \theta = \frac{1}{2}$$

$$\text{or } \theta = 30^\circ$$



19. (2) Since $f(x) = \sin x$ is an increasing function for $0 < x < \pi/2$ and 1 rad is approximately 57° , we have

$$1^\circ < 1^R \Rightarrow \sin 1^\circ < \sin 1$$

$$20. (1) \text{ Now, } \sin^2 \theta = \frac{x^2 + y^2}{2xy}$$

$$\therefore x, y \text{ have same sign}$$

$$\text{Now } \frac{x^2 + y^2}{2xy} = \frac{1}{2} \left[\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} \right)^2 + 2 \right] \geq 1$$

$$\text{But } \sin^2 \theta \leq 1. \text{ Therefore, } \frac{x^2 + y^2}{2xy} = 1$$

$$\Rightarrow x = y.$$

21. (3) $\sin^2 \theta \leq 1$

$$\Rightarrow \frac{x^2 + y^2 + 1}{2x} \leq 1$$

$$\text{or } x^2 + y^2 - 2x + 1 \leq 0$$

$$\text{or } (x-1)^2 + y^2 \leq 0$$

It is possible, iff $x-1=0$ and $y=0$.

(as $x > 0$)

22. (2) Given, $\sec^2 \theta = \frac{4xy}{(x+y)^2}$

Now $\sec^2 \theta \geq 1 \Rightarrow \frac{4xy}{(x+y)^2} \geq 1$

or $(x+y)^2 \leq 4xy$

or $(x+y)^2 - 4xy \leq 0$

or $(x-y)^2 \leq 0$

But for real values of x and y ,

$(x-y)^2 \geq 0$ or $(x-y)^2 = 0$

$\therefore x = y$

Also $x + y \neq 0 \Rightarrow x \neq 0, y \neq 0$

23. (4) The given relation is satisfied only when

$\sin \theta_1 = \sin \theta_2 = \sin \theta_3 = 1$

$\Rightarrow \cos \theta_1 = \cos \theta_2 = \cos \theta_3 = 0$

$\Rightarrow \cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0$

24. (3) $\sin x + \sin y + \sin z + \sin w = -4$

$\Rightarrow \sin x = \sin y = \sin z = \sin w = -1$ (only)

So, $\sin^{400} x + \sin^{300} y + \sin^{200} z + \sin^{100} w$
 $= (-1)^{400} + (-1)^{300} + (-1)^{200} + (-1)^{100} = 4$

25. (3) For $\alpha = -\pi/2$, $\beta = -\pi/2$ and $\gamma = 2\pi$

$\sin \alpha + \sin \beta + \sin \gamma = -2$

Hence, the minimum value of the expression is negative.

26. (4) Since $0 < x < \pi$. Therefore, $\sin x > 0$

We have $1 + \sin x + \sin^2 x + \dots \infty = 4 + 2\sqrt{3}$

$\Rightarrow \frac{1}{1 - \sin x} = 4 + 2\sqrt{3}$ (sum of infinite G.P.)

or $\sin x = 1 - \frac{1}{4 + 2\sqrt{3}} = \frac{3 + 2\sqrt{3}}{4 + 2\sqrt{3}} = \frac{\sqrt{3}}{2}$

$\Rightarrow x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

27. (1) $(2 \sin^2 91^\circ - 1)(2 \sin^2 92^\circ - 1) \dots (2 \sin^2 180^\circ - 1)$,

In this product there exists a factor $(2 \sin^2 135^\circ - 1)$, which is equal to zero

Thus, the product of all terms is zero.

28. (2) $\sin A = \sin^2 B$ and $2 \cos^2 A = 3 \cos^2 B$

$\Rightarrow 2 - 2 \sin^2 A = 3 - 3 \sin^2 B$

$\Rightarrow 2 \sin^2 A - 3 \sin A + 1 = 0$

$\Rightarrow (2 \sin A - 1)(\sin A - 1) = 0$

$\Rightarrow A = 30^\circ$ or $A = 90^\circ$

If $A = 30^\circ \Rightarrow B = 45^\circ \Rightarrow C = 105^\circ$

If $A = 90^\circ \Rightarrow B = 90^\circ$, which is not possible

29. (1) $\cos \theta + \sin \theta = \frac{1}{5}$

$\Rightarrow 1 + \tan \theta = \frac{1}{5} \sec \theta$ (dividing by $\cos \theta$)

$\Rightarrow 25 + 50 \tan \theta + 25 \tan^2 \theta = \sec^2 \theta$

or $12 \tan^2 \theta + 25 \tan \theta + 12 = 0$

or $12 \tan^2 \theta + 16 \tan \theta + 9 \tan \theta + 12 = 0$

or $(4 \tan \theta + 3)(3 \tan \theta + 4) = 0$

$\Rightarrow \tan \theta = -\frac{3}{4}$ or $\tan \theta = -\frac{4}{3}$

$\tan \theta = -\frac{3}{4}$ is rejected

as then $\cos \theta = -\frac{4}{5}$ and $\sin \theta = \frac{3}{5}$

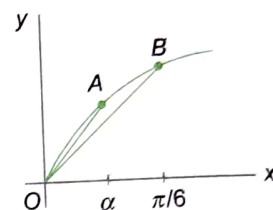
for which $\cos \theta + \sin \theta = -\frac{1}{5}$

Hence, $\tan \theta = -\frac{4}{3}$

30. (2) $\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} + \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = \frac{1 - \cos \alpha + 1 + \cos \alpha}{\sqrt{1 - \cos^2 \alpha}}$
 $= \frac{2}{|\sin \alpha|} = \frac{2}{-\sin \alpha}$ ($\because \pi < \alpha < 3\pi/2$)

31. (3) In the graph of $y = \sin x$. Let

$A \equiv (\alpha, \sin \alpha)$, $B = \left(\frac{\pi}{6}, \sin \frac{\pi}{6}\right)$



Clearly, slope of $OA >$ slope of OB , so

$\frac{\sin \alpha}{\alpha} > \frac{\sin \pi/6}{\pi/6} = \frac{3}{\pi} \Rightarrow \frac{\alpha}{\sin \alpha} < \frac{\pi}{3}$

32. (2) $2 \sin^2 \theta + 3 \cos^2 \theta = 2(\sin^2 \theta + \cos^2 \theta) + \cos^2 \theta$
 $= 2 + \cos^2 \theta \geq 2$ ($\because \cos^2 \theta > 0$)

33. (2) $\sin^4 \theta + \cos^4 \theta = (\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta$
 $= 1 - 2 \sin^2 \theta \cos^2 \theta \leq 1$

34. (1) $f(x) = \cos^6 x + \sin^6 x$
 $= (\cos^2 x + \sin^2 x)^3 - 3 \cos^2 x \sin^2 x (\cos^2 x + \sin^2 x)$
 $= 1 - 3 \cos^2 x (1 - \cos^2 x)$
 $= 3 \cos^4 x - 3 \cos^2 x + 1$
 $= 3 \left(\cos^4 x - \cos^2 x + \frac{1}{3} \right)$
 $= 3 \left(\left(\cos^2 x - \frac{1}{2} \right)^2 + \frac{1}{12} \right)$

Least value of $f(x)$ is $\frac{1}{4}$, when $\cos^2 x - \frac{1}{2} = 0$

Greatest value of $f(x)$ is 1, when $\cos^2 x = 0$ or 1

35. (2) Minimum value of $a \tan^2 x + b \cot^2 x$ is $2\sqrt{ab}$.

And maximum value of $a \sin^2 \theta + b \cos^2 \theta$ is a .

($\because a \sin^2 \theta + b \cos^2 \theta = (a - b) \sin^2 \theta + b$)

According to question, $a = 2\sqrt{ab}$

$\therefore a = 4b$

36. (4) $f(\theta) = \cos^2 \theta (\cos^2 \theta + 1) + 2 \sin^2 \theta$
 $= \cos^4 \theta + \cos^2 \theta + \sin^2 \theta + \sin^2 \theta$
 $= \cos^4 \theta + 1 + 1 - \cos^2 \theta$
 $= \left(\cos^2 \theta - \frac{1}{2} \right)^2 + 2 - \frac{1}{4}$
 $= \left(\cos^2 \theta - \frac{1}{2} \right)^2 + \frac{7}{4}$

$$f_{\min.} = \frac{7}{4}$$

$$f_{\max.} = \frac{1}{4} + \frac{7}{4} = 2$$

(1) Using A.M. $\geq$ G.M.

$$\frac{\sin \theta + \sin \theta + \sin \theta + \operatorname{cosec}^3 \theta}{4} \geq (\sin^3 \theta \cdot \operatorname{cosec}^3 \theta)^{1/4}$$

Hence, minimum value of $3 \sin \theta + \operatorname{cosec}^3 \theta$ is 4.

(2) We have,

$$0 < i < \pi, i = 1, 2, 3, \dots, n$$

$$\sin \theta_i > 0, i = 1, 2, 3, \dots, n$$

$$\therefore \left(\frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3 + \dots + \sin \theta_n}{n} \right)$$

$$\leq \sin \left(\frac{\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n}{n} \right) = \sin \left(\frac{\pi}{n} \right)$$

$$\therefore \sin \theta_1 + \sin \theta_2 + \sin \theta_3 + \dots + \sin \theta_n \leq n \sin \left(\frac{\pi}{n} \right)$$

(3) We have

$$\tan^2 \theta + \sec \theta = \lambda$$

$$\Rightarrow \sec^2 \theta + \sec \theta - (\lambda + 1) = 0$$

$$\Rightarrow \sec \theta = \frac{-1 \pm \sqrt{4\lambda + 5}}{2}$$

For real sec θ ,

$$4\lambda + 5 \geq 0 \Rightarrow \lambda \geq \frac{-5}{4}$$

...(1)

Also, sec $\theta \geq 1$ or sec $\theta \leq -1$

$$\Rightarrow \frac{-1 \pm \sqrt{4\lambda + 5}}{2} \geq 1 \text{ or } \frac{-1 \pm \sqrt{4\lambda + 5}}{2} \leq -1$$

$$\Rightarrow 4\lambda + 5 \geq 9 \text{ or } 4\lambda + 5 \geq 1$$

$$\Rightarrow \lambda \geq 1 \text{ or } \lambda \geq -1$$

...(2)

$\therefore$ From (1) and (2), we get $\lambda \in [-1, \infty)$.

(4) We know that $0 \leq \sin^2 \theta \leq 1$

$$\Rightarrow \sin^8 \theta \leq \sin^2 \theta$$

$$\text{Similarly, } \cos^{14} \theta \leq \cos^2 \theta$$

On adding, we get

$$\sin^8 \theta + \cos^{14} \theta \leq \sin^2 \theta + \cos^2 \theta$$

$$\text{or } \sin^8 \theta + \cos^{14} \theta < 1$$

$$(5) y = 256 \sin^2 x + 324 \operatorname{cosec}^2 x$$

$$= (16 \sin x - 18 \operatorname{cosec} x)^2 + 576$$

$$\geq 576$$

$$(6) \text{ Let } s = a \sec \theta - b \tan \theta$$

$$\text{or } b \tan \theta + s = a \sec \theta$$

$$\text{or } (a^2 - b^2) \tan^2 \theta - 2bs \tan \theta + (a^2 - s^2) = 0$$

$$\text{For } \tan \theta \text{ to be real } 4b^2 s^2 - 4(a^2 - b^2)(a^2 - s^2) \geq 0$$

$$\text{or } a^2 s^2 \geq a^2(a^2 - b^2)$$

$$\text{or } s \geq \sqrt{a^2 - b^2}$$

Therefore, the minimum value of s is $\sqrt{a^2 - b^2}$.

$$(7) y = (\sin^2 x + \cos^2 x) + 2(\sin x \operatorname{cosec} x + \cos x \sec x) + \sec^2 x + \operatorname{cosec}^2 x$$

$$= 5 + 2 + \tan^2 x + \cot^2 x$$

$$= 7 + (\tan x - \cot x)^2 + 2$$

$$\therefore y_{\min} = 9$$

$$44. (4) |\sin x \cos x| + |\tan x + \cot x| = \sqrt{3}$$

$$\text{or } |\sin x \cos x| + \frac{1}{|\sin x \cos x|} = \sqrt{3}$$

$$\text{But } |\sin x \cos x| + \frac{1}{|\sin x \cos x|} \geq 2$$

Hence, no solution.

$$45. (4) \cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$$

$$\text{or } \cot^4 x - 2 \cot^2 x + a^2 - 2 = 0$$

$$\text{or } (\cot^2 x - 1)^2 = 3 - a^2$$

To have at least one solution, $3 - a^2 \geq 0$

$$\text{or } a^2 - 3 \leq 0$$

$$a \in [-\sqrt{3}, \sqrt{3}]$$

Integral values are $-1, 0, 1$; therefore, the sum is 0.

$$46. (2) \cos^2 x - (c - 1) \cos x + 2c \geq 6 \quad \forall x \in R$$

$$\Rightarrow (\cos x - 2)(\cos x - \forall x \in R)$$

$$\Rightarrow \cos x \leq c - 3$$

$$\Rightarrow c - 3 \geq 1$$

$$\Rightarrow c \in [4, \infty)$$

$$47. (2) \sin^2 x + a \cos x + a^2 > 1 + \cos x$$

Putting $x = 0$, we have

$$a + a^2 > 2$$

$$\text{or } a^2 + a - 2 > 0$$

$$\text{or } (a + 2)(a - 1) > 0$$

$$\text{or } a < -2 \text{ or } a > 1$$

Therefore, the largest negative integral value of a is -3 .

$$48. (2) \sqrt{2 \cot \alpha + \frac{1}{\sin^2 \alpha}} = \sqrt{2 \cot \alpha + \operatorname{cosec}^2 \alpha}$$

$$= \sqrt{2 \cot \alpha + 1 + \cot^2 \alpha}$$

$$= |1 + \cot \alpha| = -1 - \cot \alpha$$

[since $\cot \alpha < -1$ when $3\pi/4 < \alpha < \pi$]

$$49. (2) \frac{\sec \theta}{\sqrt{1 + \tan^2 \theta}} + \frac{\operatorname{cosec} \theta}{\sqrt{1 + \cot^2 \theta}}, \theta \in (\pi, 3\pi/2)$$

$$= \sec \theta |\cos \theta| + \operatorname{cosec} \theta |\sin \theta|$$

$$= \sec \theta (-\cos \theta) + \operatorname{cosec} \theta (-\sin \theta) \quad [\text{As } \theta \in (\pi, 3\pi/2)]$$

$$= -1 - 1$$

$$= -2$$

$$50. (2) f(x) = \frac{\sin x}{\sqrt{1 - \cos^2 x}} + \frac{\cos x}{\sqrt{1 - \sin^2 x}}$$

$$+ \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$$

$$= \frac{\sin x}{|\sin x|} + \frac{\cos x}{|\cos x|} + \frac{\tan x}{|\tan x|} + \frac{\cot x}{|\cot x|}$$

$$= \begin{cases} 4, & x \in \text{1st quadrant} \\ -2, & x \in \text{2nd quadrant} \\ 0, & x \in \text{3rd quadrant} \\ -2, & x \in \text{4th quadrant} \end{cases}$$

$$f(x)_{\min.} = -2$$

$$51. (2) \text{ Let } u = \cos \theta \left\{ \sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha} \right\}$$

$$\text{or } (u - \sin \theta \cos \theta)^2 = \cos^2 \theta (\sin^2 \theta + \sin^2 \alpha)$$

$$\text{or } u^2 \tan^2 \theta - 2u \tan \theta + u^2 - \sin^2 \alpha = 0$$

Since $\tan \theta$ is real, we have

$$4u^2 - 4u^2(u^2 - \sin^2 \alpha) \geq 0$$

$$\text{or } u^2 \leq 1 + \sin^2 \alpha$$

$$\text{or } |u| \leq \sqrt{1 + \sin^2 \alpha}$$

52. (4) In the second quadrant, $\sin x < \cos x$ is false, as $\sin x$ is positive and $\cos x$ is negative.

In the fourth quadrant, $\cos x < \tan x$ is false, as $\cos x$ is positive and $\tan x$ is negative.

In the third quadrant, i.e., $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ if $\tan x < \cot x$ then $\tan^2 x < 1$, which is false.

Now, $\sin x < \cos x$ is true in $\left(0, \frac{\pi}{4}\right)$ and $\tan x < \cot x$ is also true.

Further, $\cos x < \cot x$, as $\cot x = \frac{\cos x}{\sin x}$ and $\sin x < 1$.

53. (3) $k \cos^2 x - k \cos x + 1 \geq 0 \quad \forall x \in (-\infty, \infty)$

$$\Rightarrow k(\cos^2 x - \cos x) + 1 \geq 0 \quad \dots(i)$$

$$\text{But } \cos^2 x - \cos x = \left(\cos x - \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$\Rightarrow -\frac{1}{4} \leq \cos^2 x - \cos x \leq 2$$

$$\text{Now, from (i), we get } 2k + 1 \geq 0 \Rightarrow k \geq -\frac{1}{2}$$

$$\text{Also, } -\frac{k}{4} + 1 \geq 0$$

$$\Rightarrow k \leq 4$$

$$\Rightarrow -\frac{1}{2} \leq k \leq 4$$

54. (2) We have

$$\begin{aligned} & \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{5\pi}{7} \\ & \quad + \cos \frac{6\pi}{7} + \cos \frac{7\pi}{7} \\ &= \left(\cos \frac{\pi}{7} + \cos \frac{6\pi}{7}\right) + \left(\cos \frac{2\pi}{7} + \cos \frac{5\pi}{7}\right) \\ & \quad + \left(\cos \frac{3\pi}{7} + \cos \frac{4\pi}{7}\right) + \cos \pi \\ &= \left(\cos \frac{\pi}{7} - \cos \frac{\pi}{7}\right) + \left(\cos \frac{2\pi}{7} - \cos \frac{2\pi}{7}\right) \\ & \quad + \left(\cos \frac{3\pi}{7} - \cos \frac{3\pi}{7}\right) + \cos \pi \\ &= \cos \pi = -1 \end{aligned}$$

$$\begin{aligned} 55. (1) \quad & \tan \frac{\pi}{3} + 2 \tan \frac{2\pi}{3} + 4 \tan \frac{4\pi}{3} + 8 \tan \frac{8\pi}{3} \\ &= \tan \frac{\pi}{3} + 2 \tan \left(\pi - \frac{\pi}{3}\right) + 4 \tan \left(\pi + \frac{\pi}{3}\right) + 8 \tan \left(3\pi - \frac{\pi}{3}\right) \\ &= \tan \frac{\pi}{3} - 2 \tan \frac{\pi}{3} + 4 \tan \frac{\pi}{3} - 8 \tan \frac{\pi}{3} \\ &= -5 \tan \frac{\pi}{3} = -5\sqrt{3} \end{aligned}$$

$$\begin{aligned} 56. (2) \quad & 3 \left[\sin^4 \left(\frac{3}{2}\pi - \alpha\right) + \sin^4(3\pi + \alpha) \right] \\ & - 2 \left[\sin^6 \left(\frac{1}{2}\pi + \alpha\right) + \sin^6(5\pi - \alpha) \right] \\ &= 3(\cos^4 \alpha + \sin^4 \alpha) - 2(\cos^6 \alpha + \sin^6 \alpha) \\ &= 3(1 - 2 \sin^2 \alpha \cos^2 \alpha) - 2[(\sin^2 \alpha + \cos^2 \alpha)^3 \\ & \quad - 3 \sin^2 \alpha \cos^2 \alpha (\sin^2 \alpha + \cos^2 \alpha)] \\ &= 3(1 - 2 \sin^2 \alpha \cos^2 \alpha) - 2[1 - 3 \sin^2 \alpha \cos^2 \alpha] \\ &= 1 \end{aligned}$$

$$\begin{aligned} 57. (2) \quad & E = \log_{10} (\tan 6^\circ) + \log_{10} (\tan 12^\circ) + \log_{10} (\tan 18^\circ) + \dots \\ & \quad + \log_{10} (\tan 84^\circ) \\ &= \log_{10} ((\tan 6^\circ \times \tan 84^\circ) (\tan 12^\circ \times \tan 78^\circ) \dots \\ & \quad (\tan 42^\circ \times \tan 48^\circ)) \\ &= \log_{10} ((\tan 6^\circ \times \cot 6^\circ) (\tan 12^\circ \times \cot 12^\circ) \dots \\ & \quad (\tan 42^\circ \times \cot 42^\circ)) \\ &= \log_{10} 1 = 0 \end{aligned}$$

Multiple Correct Answers Type

1. (1), (2), (3), (4)

$$\cos^2 \theta = 1 - \sin^2 \theta.$$

Let $81^{\sin^2 \theta} = z$, we get

$$z + \frac{81}{z} = 30$$

$$\text{or } z^2 - 30z + 81 = 0$$

$$\text{or } (z - 27)(z - 3) = 0$$

$$\text{i.e., } 81^{\sin^2 \theta} = 3^4 \sin^2 \theta = 3^3 \text{ or } 3^1$$

$$\therefore \sin^2 \theta = \frac{3}{4}, \frac{1}{4}$$

$$\text{or } \sin \theta = \pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2}$$

$$\therefore \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ$$

2. (2), (3), (4)

Opposite angles of a cyclic quadrilateral are supplementary.

3. (1), (2), (3), (4)

$$(1) \text{ For } x \in \left(0, \frac{\pi}{4}\right), \tan x < \cot x$$

$$\text{Also } \ln(\sin x) < 0$$

$$\Rightarrow (\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}$$

$$(2) \text{ For } x \in \left(0, \frac{\pi}{2}\right), \operatorname{cosec} x \geq 1$$

$$\Rightarrow \ln(\operatorname{cosec} x) \geq 0$$

$$\Rightarrow 4^{\ln(\operatorname{cosec} x)} < 5^{\ln(\operatorname{cosec} x)}$$

$$(3) x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \cos x \in (0, 1)$$

$$\Rightarrow \ln(\cos x) < 0$$

$$\text{Also } \frac{1}{2} > \frac{1}{3}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{\ln(\cos x)} < \left(\frac{1}{3}\right)^{\ln(\cos x)}$$

$$(4) \text{ For } x \in \left(0, \frac{\pi}{2}\right)$$

Since $\sin x < \tan x$, we get $\ln(\sin x) < \ln(\tan x)$

$$\Rightarrow 2^{\ln(\sin x)} < 2^{\ln(\tan x)}$$

4. (1), (4)

We have $\tan A = -\frac{4}{3}$ If A lies in 2nd quadrant, then

$$\sin A = \frac{4}{5}, \cos A = \frac{-3}{5}, \text{ and } \cot A = \frac{-3}{4}$$

If A lies in 4th quadrant, then

$$\sin A = \frac{-4}{5}, \cos A = \frac{3}{5} \text{ and } \cot A = \frac{-3}{4}$$

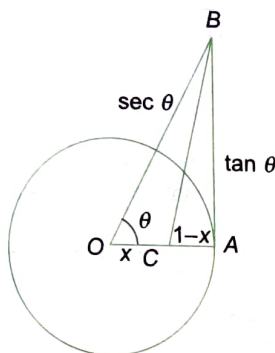
5. (1), (3), (4)

Using property of angle bisector, we get

$$\frac{OB}{AB} = \frac{OC}{AC} \Rightarrow \frac{\sec \theta}{\tan \theta} = \frac{x}{1-x}$$

$$\text{or } x = \frac{\sec \theta}{\sec \theta + \tan \theta}$$

$$= \sec \theta (\sec \theta - \tan \theta) = \frac{1}{1 + \sin \theta}$$



6. (2), (4)

Divide by $\cos \alpha$ and square both sides and let $\tan \alpha = t$ so that $\sec^2 \alpha = 1 + t^2$

$$\Rightarrow [(a+2)t + (2a-1)]^2 = [(2a+1)^2 (1+t^2)]$$

$$\text{or } t^2[(a+2)^2 - (2a+1)^2] + 2(a+2)(2a-1)t + [(2a-1)^2 - (2a+1)^2] = 0$$

$$\text{or } 3(1-a^2)t^2 + 2(2a^2+3a-2)t - 4 \times 2a = 0$$

$$\text{or } 3(1-a^2)t^2 - 4(1-a^2)t + 6at - 8a = 0$$

$$\text{or } t(1-a^2)(3t-4) + 2a(3t-4) = 0$$

$$\text{or } (3t-4)[(1-a^2)t + 2a] = 0$$

$$\text{or } t = \tan \alpha = \frac{4}{3} \text{ or } \frac{2a}{a^2-1}$$

7. (1), (2), (3)

$$\log_{1/3} \log_7(\sin x + a) > 0$$

$$\text{or } 0 < \log_7(\sin x + a) < 1$$

$$\text{or } 1 < (\sin x + a) < 7, \forall x \in R$$

$$\text{or } 1 - \sin x < a < 7 - \sin x$$

It is found that a should be less than the minimum value of $(7 - \sin x)$ and a must be greater than the maximum value of $(1 - \sin x)$. Thus,

$$\Rightarrow 2 < a < 6$$

8. (1), (3)

$$\log_b \sin t = x \text{ or } \sin t = b^x$$

$$\text{Let } \log_b(\cos t) = y, \text{ then } b^y = \cos t$$

$$\text{or } b^{2y} = \cos^2 t = 1 - \sin^2 t = 1 - b^{2x}$$

$$\text{or } 2y = \log_b(1 - b^{2x})$$

$$\text{or } y = \frac{1}{2} \log_b(1 - b^{2x}) = \log_b \sqrt{1 - b^{2x}}$$

9. (1), (2), (3)

$$(1) A = B = C = \frac{\pi}{3}$$

(2) Right angled triangle

$$(3) A = B = \frac{\pi}{4}, C = \frac{\pi}{2}$$

(4) Not possible as $\sin A, \sin B, \sin C > 0$

10. (1), (4)

$$2 \sec^2 \alpha - \sec^4 \alpha - 2 \operatorname{cosec}^2 \alpha + \operatorname{cosec}^4 \alpha = \frac{15}{4}$$

$$\text{or } 2(\sec^2 \alpha - \operatorname{cosec}^2 \alpha) + (\operatorname{cosec}^2 \alpha + \sec^2 \alpha) \times (\operatorname{cosec}^2 \alpha - \sec^2 \alpha) = \frac{15}{4}$$

$$\text{or } (\operatorname{cosec}^2 \alpha - \sec^2 \alpha) [\operatorname{cosec}^2 \alpha + \sec^2 \alpha - 2] = \frac{15}{4}$$

$$\text{or } 4(\cot^2 \alpha - \tan^2 \alpha) (\cot^2 \alpha + \tan^2 \alpha) = 15$$

$$\text{or } 4(\cot^4 \alpha - \tan^4 \alpha) = 15$$

$$\text{or } 4(1 - \tan^8 \alpha) = 15 \tan^4 \alpha$$

$$\text{or } 4 \tan^8 \alpha + 15 \tan^4 \alpha - 4 = 0$$

$$\text{or } (4 \tan^4 \alpha - 1) (\tan^4 \alpha + 4) = 0$$

$$\text{or } \tan^4 \alpha = \frac{1}{4} \text{ or } \tan^4 \alpha = -4$$

$$\text{or } \tan^2 \alpha = \pm \frac{1}{2}$$

$$\text{or } \tan^2 \alpha = +\frac{1}{2}$$

$$\text{or } \tan \alpha = \pm \frac{1}{\sqrt{2}}$$

11. (1), (2), (4)

$$\cot \theta + \tan \theta = x$$

$$\text{or } \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = x$$

$$\text{or } 1 = x \sin \theta \cos \theta$$

$$\text{Now } \sec \theta - \cos \theta = y$$

$$\frac{1}{\cos \theta} - \cos \theta = y$$

$$\text{or } \sin^2 \theta = y \cos \theta$$

$$\text{Now } x^2 y = \frac{1}{\sin^2 \theta \cos^2 \theta} \cdot \frac{\sin^2 \theta}{\cos \theta} = \frac{1}{\cos^3 \theta}$$

$$\text{and } xy^2 = \frac{\sin^3 \theta}{\cos^3 \theta}$$

$$\therefore (x^2 y)^{2/3} - (xy^2)^{2/3} = \frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} = 1$$

12. (2), (3), (4)

$$\text{We have } x = \frac{1 - \sin \phi}{\cos \phi}, y = \frac{1 + \cos \phi}{\sin \phi}$$

Multiplying, we get

$$xy = \frac{(1 - \sin \phi)(1 + \cos \phi)}{\cos \phi \sin \phi}$$

$$\Rightarrow xy + 1 = \frac{1 - \sin\phi + \cos\phi - \sin\phi \cos\phi + \sin\phi \cos\phi}{\cos\phi \sin\phi}$$

$$= \frac{1 - \sin\phi + \cos\phi}{\cos\phi \sin\phi}$$

and $x - y = \frac{(1 - \sin\phi)\sin\phi - \cos\phi(1 + \cos\phi)}{\cos\phi \sin\phi}$

$$= \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi}{\cos\phi \sin\phi}$$

$$= \frac{\sin\phi - \cos\phi - 1}{\cos\phi \sin\phi} = -(xy + 1)$$

Thus, $xy + x - y + 1 = 0$, $x = \frac{y-1}{y+1}$, and $y = \frac{1+x}{1-x}$.

13. (1), (2), (3)

$$f(\alpha) = \sqrt{\operatorname{cosec}^2 \alpha - 2 \cot \alpha} + \sqrt{\operatorname{cosec}^2 \alpha + 2 \cot \alpha}$$

$$= \sqrt{1 + \cot^2 \alpha - 2 \cot \alpha} + \sqrt{1 + \cot^2 \alpha + 2 \cot \alpha}$$

$$= |\cot \alpha - 1| + |\cot \alpha + 1|$$

Case I: $\cot \alpha \leq -1$

$$\therefore f(\alpha) = -\cot \alpha + 1 - \cot \alpha - 1 = -2 \cot \alpha$$

Case II: $-1 \leq \cot \alpha \leq 1$

$$\therefore f(\alpha) = -\cot \alpha + 1 + \cot \alpha + 1 = 2$$

Case III: $\cot \alpha \geq 1$

$$\therefore f(\alpha) = (\cot \alpha - 1) + (\cot \alpha + 1) = 2 \cot \alpha$$

14. (1), (2)

We have

$$\frac{y+3}{2y+5} = \sin^2 x + 2 \cos x + 1$$

$$\therefore \cos^2 x - 2 \cos x + 1 = 3 - \frac{y+3}{2y+5}$$

$$\therefore (\cos x - 1)^2 = \frac{5y+12}{2y+5}$$

Now $-1 \leq \cos x \leq 1$

$$\therefore -2 \leq \cos x - 1 \leq 0$$

$$\therefore 0 \leq (\cos x - 1)^2 \leq 4$$

$$\therefore 0 \leq \frac{5y+12}{2y+5} \leq 4$$

$$\therefore 0 \leq \frac{5y+12}{2y+5} \text{ and } \frac{5y+12}{2y+5} \leq 4$$

$$\therefore \frac{5y+12}{2y+5} \geq 0 \text{ and } \frac{3y+8}{2y+5} \geq 0$$

$$\therefore \left(-\infty, -\frac{5}{2}\right) \cup \left[-\frac{12}{5}, \infty\right) \text{ and } \left(-\infty, -\frac{8}{3}\right) \cup \left[-\frac{5}{2}, \infty\right)$$

$$\therefore y \in \left(-\infty, -\frac{8}{3}\right] \cup \left[-\frac{12}{5}, \infty\right)$$

15. (1), (2), (3), (4)

$$\cos \alpha = \frac{1}{2} \left(x + \frac{1}{x}\right) \text{ and } \cos \beta = \frac{1}{2} \left(y + \frac{1}{y}\right)$$

since $xy > 0$, we have

$$x + \frac{1}{x} \geq 2 \text{ or } \leq -2 \text{ and } y + \frac{1}{y} \geq 2 \text{ or } \leq -2$$

$$\Rightarrow \cos \alpha = 1, \cos \beta = 1$$

$$\text{or } \cos \alpha = -1, \cos \beta = -1$$

$$\therefore \cos \alpha \cos \beta = 1$$

$$\Rightarrow \alpha + \beta \text{ is an even multiple of } \pi$$

$$(\cos \alpha + \cos \beta)^2 = 4$$

$$\Rightarrow \sin(\alpha + \beta + \gamma) = \sin(2n\pi + \gamma) = \sin \gamma$$

$$\text{Also, } \sin \alpha = \sin \beta = 0.$$

16. (1), (3), (4)

$$n_1 = \sin 15^\circ - \cos 15^\circ < -ve$$

$$(\cos 15^\circ > \sin 15^\circ)$$

$$n_2 = \cos 93^\circ + \sin 93^\circ$$

$$= -\sin 3^\circ + \cos 3^\circ > 0$$

$$(\cos 3^\circ > \sin 3^\circ)$$

$$n_3 = \tan 27^\circ - \cot 27^\circ < 0$$

$$(\tan 27^\circ < \cot 27^\circ)$$

$$n_4 = \cot 127^\circ + \tan 127^\circ < 0$$

$$(\tan 127^\circ, \cot 127^\circ < 0).$$

17. (2), (3)

All are infinite geometric progression with common ratio < 1 .

$$x = \frac{1}{1 - \cos^2 \phi} = \frac{1}{\sin^2 \phi}, y = \frac{1}{1 - \sin^2 \phi} = \frac{1}{\cos^2 \phi},$$

$$z = \frac{1}{1 - \cos^2 \phi \sin^2 \phi}$$

$$\text{Now, } xy + z = \frac{1}{\sin^2 \phi \cos^2 \phi} + \frac{1}{1 - \sin^2 \phi \cos^2 \phi}$$

$$= \frac{1}{\sin^2 \phi \cos^2 \phi (1 - \sin^2 \phi \cos^2 \phi)}$$

$$\text{or } xy + z = xyz$$

...(i)

$$\text{Clearly, } x + y = \frac{\sin^2 \phi + \cos^2 \phi}{\sin^2 \phi \cos \phi} = xy$$

$$\therefore x + y + z = xyz$$

[using Eq. (i)]

Linked Comprehension Type

1. (3), 2. (2), 3. (2)

$$\text{We have, } \frac{\cos^4 x}{a} + \frac{\sin^4 x}{b} = \frac{1}{a+b} = \frac{\cos^2 x + \sin^2 x}{a+b}$$

$$\Rightarrow \cos^2 x \left(\frac{\cos^2 x}{a} - \frac{1}{a+b} \right) = \sin^2 x \left(\frac{1}{a+b} - \frac{\sin^2 x}{b} \right)$$

$$\Rightarrow \cos^2 x \left(\frac{b \cos^2 x - a \sin^2 x}{a(a+b)} \right) = \sin^2 x \left(\frac{b \cos^2 x - a \sin^2 x}{b(a+b)} \right)$$

$$\Rightarrow \frac{\cos^2 x}{a} = \frac{\sin^2 x}{b}$$

$$\Rightarrow \frac{1 - \sin^2 x}{a} = \frac{\sin^2 x}{b}$$

$$\Rightarrow \sin^2 x = \frac{b}{a+b} \text{ and } \cos^2 x = \frac{a}{a+b}$$

$$\therefore \frac{\sin^8 x}{b^3} + \frac{\cos^8 x}{a^3} = \frac{b^4}{b^3(a+b)^4} + \frac{a^4}{a^3(a+b)^4} = \frac{1}{(a+b)^3}$$

4. (1), 5. (2), 6. (3)

$$0 < \alpha < 90^\circ, 90^\circ < \beta < 180^\circ,$$

$$180^\circ < \gamma < 270^\circ, 270^\circ < \delta < 360^\circ$$

$$\Rightarrow \gamma < 270^\circ < \alpha + \delta < 450^\circ$$

$$\Rightarrow \alpha + \delta \text{ lies in the I or IV quadrant and cosine in both is positive.}$$

If d is the common ratio of the A.P., then
 $\beta = \alpha + d, \gamma = \alpha + 2d, \delta = \alpha + 3d$

$$\Rightarrow \beta + \gamma = \alpha + \delta, 2(\alpha - \beta) = -2d = \beta - \delta$$

$$\text{and } \alpha + \gamma = 2\beta;$$

$$\text{Now, } \beta - \gamma = -d, \alpha - \delta = -3d$$

$$270^\circ < \delta < 360^\circ$$

$$\Rightarrow 270^\circ < \alpha + 3d < 360^\circ$$

$$\Rightarrow 200^\circ < 3d < 290^\circ$$

$$\Rightarrow 400^\circ < 6d < 580^\circ$$

$$\Rightarrow 680^\circ < 4\alpha + 6d < 860^\circ$$

$$\Rightarrow 680^\circ < \theta < 860^\circ$$

$$(\therefore \alpha = 70^\circ)$$

8. (1)

In given $\triangle ABC$ both $\frac{A}{2}$ and $\frac{B}{2}$ lie strictly in $(0, \frac{\pi}{2})$ and

$\sin x$ is always increasing in $(0, \frac{\pi}{2})$ whereas $\cos x$ is always decreasing in $(0, \frac{\pi}{2})$.

$$\text{So, if } \frac{A}{2} > \frac{B}{2}$$

$$\Rightarrow \sin \frac{A}{2} > \sin \frac{B}{2}$$

$$\text{or } x_1 > x_2$$

$$\text{and } x_3 < x_4$$

$$\text{or } \frac{1}{x_3} > \frac{1}{x_4}$$

$$\text{So, } x_1^{2007} x_4^{2006} = x_2^{2007} x_3^{2006} \text{ is not valid.}$$

$$\text{Similarly for } \frac{A}{2} < \frac{B}{2}$$

$$\Rightarrow \sin \frac{A}{2} < \sin \frac{B}{2}$$

$$\Rightarrow x_1 < x_2$$

$$\text{and } \frac{1}{x_3} < \frac{1}{x_4}$$

$$\text{For this also } x_1^{2007} x_4^{2006} = x_2^{2007} x_3^{2006} \text{ is not valid.}$$

$$\text{So } \left(\frac{x_1}{x_2}\right)^{2007} - \left(\frac{x_3}{x_4}\right)^{2006} = 0 \text{ is possible only when } \frac{A}{2} = \frac{B}{2}.$$

$$\Rightarrow x_1 = x_2 \text{ and } \frac{1}{x_3} = \frac{1}{x_4}$$

Hence, $\triangle ABC$ is isosceles with $\angle ABC = \angle CAB$.

$$\Rightarrow BC = AC = 1 \text{ unit}$$

$$\text{If } \angle A = 90^\circ;$$

$$\text{Area, } A = \frac{1}{2} BC \times AC = \frac{1}{2} \text{ sq. units}$$

$$9. (4) f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$$

$$f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$$

$$= (\sin^2 x)^3 + (\cos^2 x)^3 + k[(\sin^2 x)^2 + (\cos^2 x)^2]$$

$$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cdot \cos^2 x (\sin^2 x + \cos^2 x)$$

$$+ k[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cdot \cos^2 x]$$

$$= (1 - 3\sin^2 x \cos^2 x) + k[1 - 2\sin^2 x \cos^2 x]$$

$$f(x) \text{ is constant if } k = -3/2.$$

$$10. (3) f(x) = 0$$

$$\Rightarrow (1 - 3\sin^2 x \cos^2 x) + k[1 - 2\sin^2 x \cos^2 x] = 0$$

$$\Rightarrow k + 1 = \frac{\sin^2 x \cos^2 x}{1 - 2\sin^2 x \cos^2 x}$$

$$\Rightarrow k = \frac{3\sin^2 x \cos^2 x - 1}{1 - 2\sin^2 x \cos^2 x}$$

$$= \frac{3}{2} \frac{1 - 2\sin^2 x \cos^2 x - \frac{1}{3}}{1 - 2\sin^2 x \cos^2 x}$$

$$= -\frac{3}{2} \left(1 - \frac{\frac{1}{3}}{1 - 2\sin^2 x \cos^2 x} \right)$$

Minimum of $\sin^2 x \cos^2 x$ is 0 at $x = 0, \pi/2$

Maximum of $\sin^2 x \cos^2 x$ is $1/4$ at $x = \pi/2$

$$\text{Hence, } k \in \left[-1, -\frac{1}{2}\right]$$

$$11. (1) (1 - 3\sin^2 x \cos^2 x) + k[1 - 2\sin^2 x \cos^2 x] = 0 \text{ is an identity,}$$

$$\text{i.e., } (1 + k) - (3 + 2k)\sin^2 x \cos^2 x = 0 \text{ is an identity.}$$

$$\Rightarrow 1 + k = 0 \text{ and } 3 + 2k = 0, \text{ which do not hold simultaneously.}$$

Matrix Match Type

$$1. a \rightarrow s; b \rightarrow s; c \rightarrow p; d \rightarrow r$$

$$a. \frac{99\pi}{2} < \theta < 50\pi \Rightarrow \theta \in 4\text{th quadrant}$$

$$\therefore |\cos \theta| = \cos \theta \text{ and } |\sin \theta| = -\sin \theta$$

$$\therefore x = -\sin^2 \theta \text{ and } y = \cos^2 \theta$$

$$\Rightarrow y - x = 1$$

$$b. E = \frac{(-\cos x)(\cos^3 x) - \cos x \cdot \sin^3 x}{-\cos x \sin x - \cos^2 x} + \frac{\tan x}{\sec^2 x}$$

$$= \frac{\cos^4 x + \cos x \sin^3 x}{\sin x \cos x + \cos^2 x} + \frac{\sin x}{\sec x}$$

$$= \frac{\cos x(\sin^3 x + \cos^3 x)}{\cos x(\sin x + \cos x)} + \sin x \cos x$$

$$= (1 - \sin x \cos x) + \sin x \cos x = 1$$

$$c. \sin(-870^\circ) + \operatorname{cosec}(-660^\circ) + \tan(-855^\circ) + 2 \cot(840^\circ) + \cos(480^\circ) + \sec(900^\circ)$$

$$= -\sin(810^\circ + 60^\circ) - \operatorname{cosec}(720^\circ - 60^\circ)$$

$$- \tan(810^\circ + 45^\circ) + 2 \cot 120^\circ + \cos 120^\circ + \sec 180^\circ$$

$$= -\frac{1}{2} + \frac{2}{\sqrt{3}} + 1 - \frac{2}{\sqrt{3}} - \frac{1}{2} - 1 = -1$$

$$d. 2 \frac{(-\sin^3 x)(\cot x)(-\sec x)(-\sec x)}{\cot x \tan^2 x \sin x} = -2$$

$$2. a \rightarrow q; b \rightarrow q; c \rightarrow p, r; d \rightarrow p, s$$

a. Since angles A, B , and C are acute angles, we have

$$A + B > \pi/2$$

$$A > \frac{\pi}{2} - B$$

$$\Rightarrow \sin A - \cos B > 0$$

$$\Rightarrow \cos B - \sin A < 0$$

$$\text{Again, } B > \frac{\pi}{2} - A$$

$$\sin B > \cos A$$

$$\sin B - \cos A > 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get that x -coordinate is -ve and y -coordinate is +ve.

Therefore, point is in 2nd quadrant only.

$$\text{b. } 2^{\sin \theta} > 1 \Rightarrow \sin \theta > 0$$

$$\Rightarrow \theta \in \text{1st or 2nd quadrant}$$

$$3^{\cos \theta} < 1 \Rightarrow \cos \theta < 0$$

$$\Rightarrow \theta \in \text{2nd or 3rd quadrant}$$

Hence, $\theta \in \text{2nd quadrant}$.

$$\text{c. } |\cos x + \sin x| = |\sin x| + |\cos x|$$

Thus, $\cos x$ and $\sin x$ must have same sign or at least one is zero.

So, $x \in \text{1st or 3rd quadrant}$.

$$\text{d. L.H.S.} = \frac{1 - \sin A}{|\cos A|} + \frac{\sin A}{\cos A} = \frac{1}{\cos A},$$

which is true only if $|\cos A| = \cos A$.

3. (4)

$$\text{a. } A = \sin^2 \theta + \cos^4 \theta$$

$$= 1 - \cos^2 \theta + \cos^4 \theta$$

$$= (\cos^2 \theta - 1/2)^2 + 3/4$$

$$0 \leq \cos^2 \theta \leq 1$$

$$\Rightarrow -1/2 \leq \cos^2 \theta - 1/2 \leq 1/2$$

$$\Rightarrow 0 \leq (\cos^2 \theta - 1/2)^2 \leq 1/4$$

$$\Rightarrow 3/4 \leq (\cos^2 \theta - 1/2)^2 + 3/4 \leq 1$$

$$\text{b. } 3 \cos^2 \theta + \sin^4 \theta = 3 - 3 \sin^2 \theta + \sin^4 \theta$$

$$= (\sin^2 \theta - 3/2)^2 + 3/4$$

$$0 \leq \sin^2 \theta \leq 1$$

$$\Rightarrow -3/2 \leq \sin^2 \theta - 3/2 \leq -1/2$$

$$\Rightarrow 1/4 \leq (\sin^2 \theta - 3/2)^2 \leq 9/4$$

$$\Rightarrow 1 \leq (\sin^2 \theta - 3/2)^2 + 3/4 \leq 3$$

$$\text{c. } A = \sin^2 \theta - \cos^4 \theta$$

$$= 1 - \cos^2 \theta - \cos^4 \theta$$

$$= 5/4 - (\cos^2 \theta + 1/2)^2$$

$$0 \leq \cos^2 \theta \leq 1$$

$$\Rightarrow 1/2 \leq \cos^2 \theta + 1/2 \leq 3/2$$

$$\Rightarrow 1/4 \leq (\cos^2 \theta + 1/2)^2 \leq 9/4$$

$$\Rightarrow -9/4 \leq -(\cos^2 \theta + 1/2)^2 \leq -1/4$$

$$\Rightarrow -1 \leq 5/4 - (\cos^2 \theta + 1/2)^2 \leq 1$$

$$\text{d. } A = \tan^2 \theta + 2 \cot^2 \theta$$

$$= (\tan \theta - \sqrt{2} \cot \theta)^2 + 2\sqrt{2} \geq 2\sqrt{2}$$

Numerical Value Type

$$\text{1. (1) } \tan^2 20^\circ - \sin^2 20^\circ = \tan^2 20^\circ (1 - \cos^2 20^\circ) \\ = \tan^2 20^\circ \sin^2 20^\circ$$

$$\text{2. (1) } \sin^2 x + \cos^2 y = \frac{3a}{2}$$

...(i)

$$\text{and } \cos^2 x + \sin^2 y = \frac{a^2}{2}$$

...(ii)

Adding (i) and (ii), we get

$$2 = \frac{3a}{2} + \frac{a^2}{2}$$

$$\text{or } a^2 + 3a - 4 = 0$$

$$\text{or } (a+4)(a-1) = 0$$

$$\therefore a = -4 \text{ (rejected), } a = 1$$

$$\text{3. (7) Let } y = \cos x - \sin x \text{ (} y > 0 \text{)}$$

$$y^2 = 1 - 2 \sin x \cdot \cos x$$

$$\cos x + \sin x = \frac{5}{4}$$

$$\therefore 1 + 2 \sin x \cos x = \frac{25}{16}$$

$$\text{or } 2 \sin x \cdot \cos x = \frac{9}{16}$$

$$\text{Now } y^2 = 1 - 2 \sin x \cos x = 1 - \frac{9}{16} = \frac{7}{16}$$

(Given)

$$\text{4. (2) } \sin^4 t + \cos^4 t - 1 = (\sin^2 t + \cos^2 t)^2 - 2 \sin^2 t \cos^2 t - 1 \\ = -2 \sin^2 t \cos^2 t$$

$$\sin^6 t + \cos^6 t - 1 = (\sin^2 t + \cos^2 t)^3 - 3 \sin^2 t \cos^2 t - 1 \\ = -3 \sin^2 t \cos^2 t$$

$$\text{Hence, } 3 \frac{\sin^4 t + \cos^4 t - 1}{\sin^6 t + \cos^6 t - 1} = 2$$

$$\text{5. (1) } \sin \theta - \cos \theta = 1$$

$$\text{or } \sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta = 1$$

$$\text{or } \sin \theta \cos \theta = 0$$

$$\text{Now } \sin^3 \theta - \cos^3 \theta = 1 (\sin^2 \theta + \cos^2 \theta + \sin \theta \cos \theta) \\ = (\sin \theta - \cos \theta)^2 + 3 \sin \theta \cos \theta \\ = 1 + 3 \sin \theta \cos \theta = 1$$

$$\text{6. (1) Given that } \tan^2 \theta = \sin \theta \cos \theta$$

$$\therefore \sin \theta = \cos^3 \theta$$

Also, given expression can be rewritten as

$$\sin^6 \theta - 3 \sin^4 \theta + 3 \sin^2 \theta - 1 + 1 + \sin^2 \theta$$

$$= (\sin^2 \theta - 1)^3 + 1 + \sin^2 \theta$$

$$= -(1 - \sin^2 \theta)^3 + 1 + \sin^2 \theta$$

$$= -\cos^6 \theta + 1 + \sin^2 \theta$$

$$= -\sin^2 \theta + 1 + \sin^2 \theta = 1$$

$$\text{7. (44.5) We have, } f(\theta) = \frac{(\sin \theta)^x}{(\cos \theta)^x + (\sin \theta)^x}$$

$$\Rightarrow f(\theta) + f\left(\frac{\pi}{2} - \theta\right) = 1.$$

$$\therefore S = \sum_{\theta=1^\circ}^{89^\circ} f(\theta) = f(1^\circ) + f(2^\circ) + \dots + f(88^\circ) + f(89^\circ)$$

$$= \left(\underbrace{1+1+1+\dots+1}_{44 \text{ times}} \right) + \frac{1}{2} = 44 + \frac{1}{2} = \frac{89}{2}$$

$$\text{8. (7) } f(x) = 9 \sin^2 x - 16 \cos^2 x - 10(3 \sin x - 4 \cos x)$$

$$- 10(3 \sin x + 4 \cos x) + 100$$

$$= 25 \sin^2 x - 60 \sin x + 84$$

$$= (5 \sin x - 6)^2 + 48$$

The minimum value of $f(x)$ occurs when $\sin x = 1$.

Therefore, the minimum value of $\sqrt{f(x)}$ is 7.

$$\text{9. (18) Using A.M. } \geq \text{G.M. (for positive numbers)}$$

$$(a^2 - a + 1) + \frac{8 \sin^2 a}{\sqrt{a^2 - a + 1}} + \frac{27 \operatorname{cosec}^2 a}{\sqrt{a^2 - a + 1}}$$

$$\geq 3 \left((a^2 - a + 1) \frac{8 \sin^2 a}{\sqrt{(a^2 - a + 1)}} \frac{27 \operatorname{cosec}^2 a}{\sqrt{(a^2 - a + 1)}} \right)^{\frac{1}{3}}$$

$$= 3 (2^3 3^3)^{\frac{1}{3}} = 3 (6) = 18$$

11. (8) Let $a = \tan^2 \alpha$, $b = \tan^2 \beta$
Given expression becomes

$$\frac{(a+1)^2}{b} + \frac{(b+1)^2}{a} \quad (a \geq 0, b \geq 0)$$

$$= \frac{a^2 + 2a + 1}{b} + \frac{b^2 + 2b + 1}{a}$$

$$= \frac{a^2}{b} + \frac{1}{b} + \frac{b^2}{a} + \frac{1}{a} + 2 \left(\frac{a}{b} + \frac{b}{a} \right)$$

$$\geq 4 \sqrt{\frac{a^2}{b} \cdot \frac{1}{b} \cdot \frac{b^2}{a} \cdot \frac{1}{a}} + 2(2)$$

$$= 4 + 4 = 8$$

11. (16) We have $p \operatorname{cosec} \theta + q \cot \theta = 2$
And $(p \operatorname{cosec} \theta)^2 - (q \cot \theta)^2 = 5$

$$\Rightarrow p \operatorname{cosec} \theta - q \cot \theta = \frac{5}{2}$$

From (1) and (2)

$$p \operatorname{cosec} \theta = \frac{9}{4}, q \cot \theta = \frac{-1}{4}$$

$$\text{Now } \operatorname{cosec}^2 \theta - \cot^2 \theta = 1$$

$$\Rightarrow \frac{81}{16p^2} - \frac{1}{16q^2} = 1$$

$$\Rightarrow 81p^{-2} - q^{-2} = 16$$

$$\Rightarrow \sqrt{81p^{-2} - q^{-2}} = 4$$

$$= \frac{\tan^3 A - 1}{(\tan A - 1) \cdot \tan A}$$

$$= \frac{\tan^2 A + \tan A + 1}{\tan A}$$

$$= \tan A + \cot A + 1$$

$$= \sec A \cdot \operatorname{cosec} A + 1$$

$$2. (4) f_4(x) - f_6(x) = \frac{1}{4} (\sin^4 x + \cos^4 x) - \frac{1}{6} (\sin^6 x + \cos^6 x)$$

$$= \frac{3(\sin^4 x + \cos^4 x) - 2(\sin^6 x + \cos^6 x)}{12}$$

$$= \frac{3(1 - 2\sin^2 x \cos^2 x) - 2(1 - 3\sin^2 x \cos^2 x)}{12}$$

$$= \frac{1}{12}$$

JEE ADVANCED

Single Correct Answer Type

$$1. (4) \text{ In set } P, \sin \theta = (\sqrt{2} + 1) \cos \theta$$

$$\text{or } \tan \theta = \sqrt{2} + 1$$

$$\text{In set } Q, (\sqrt{2} - 1) \sin \theta = \cos \theta$$

$$\text{or } \tan \theta = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$$

$$\Rightarrow P = Q$$

Multiple Correct Answers Type

1. (1), (2)

$$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$$

$$3 \sin^4 x + 2(1 - \sin^2 x)^2 = \frac{6}{5}$$

$$\Rightarrow 25 \sin^4 x - 20 \sin^2 x + 4 = 0$$

$$\Rightarrow \sin^2 x = \frac{2}{5}$$

$$\therefore \cos^2 x = \frac{3}{5}$$

$$\therefore \tan^2 x = \frac{2}{3}$$

$$\text{and } \frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$$

Archives

JEE MAIN

Single Correct Answer Type

$$1. (2) \frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\tan^2 A}{\tan A - 1} + \frac{1}{\tan A - \tan^2 A}$$

$$= \frac{1}{\tan A - 1} \left[\tan^2 A - \frac{1}{\tan A} \right]$$

Chapter 3

Concept Application Exercises

Exercise 3.1

1. Given $(\cos A + \sin A)(\cos B + \sin B) = 2$

$$\Rightarrow \cos(A - B) + \sin(A + B) = 2$$

$$\Rightarrow \cos(A - B) = 1; \sin(A + B) = 1$$

$$\Rightarrow A = B; A + B = \frac{\pi}{2}$$

$$\Rightarrow C = \frac{\pi}{2}$$

2. $2x = \tan 20^\circ + \tan 50^\circ$

$$= \frac{\sin(20^\circ + 50^\circ)}{\cos 20^\circ \cos 50^\circ}$$

$$= \frac{\sin 70^\circ}{\sin 70^\circ \sin 40^\circ} = \frac{1}{\sin 40^\circ} \quad \dots(i)$$

$$2y = \tan 20^\circ + \tan 70^\circ$$

$$= \frac{\sin(20^\circ + 70^\circ)}{\cos 20^\circ \cos 70^\circ}$$

$$= \frac{2}{2 \cos 20^\circ \sin 20^\circ} = \frac{2}{\sin 40^\circ} \quad \dots(ii)$$

$$\therefore y = \frac{1}{\sin 40^\circ} = 2x \quad [\text{by Eqs. (i) and (ii)}]$$

3. $\cos 15^\circ(\sin 75^\circ + \cos 45^\circ) + \sin 15^\circ(\cos 75^\circ - \sin 45^\circ)$

$$= \sin(75^\circ + 15^\circ) + \cos(45^\circ + 15^\circ) = 1 + \frac{1}{2} = \frac{3}{2}$$

4. $\cos \alpha \cos \beta - \sin \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta = 0$

$$\Rightarrow \cos \alpha(\cos \beta - \sin \beta) + \sin \alpha(\cos \beta - \sin \beta) = 0$$

$$\Rightarrow (\cos \beta - \sin \beta)(\cos \alpha + \sin \alpha) = 0$$

If $\cos \beta - \sin \beta = 0$, then $\tan \beta = 1$, which is not possible.

$$\therefore \sin \alpha + \cos \alpha = 0$$

$$\therefore \tan \alpha = -1$$

5. Squaring and adding, we get

$$1 + 1 + 2 \sin A \cos 2A + 2 \cos A \sin 2A$$

$$= (1/4) + (1/9) = (13/36)$$

$$\Rightarrow \sin 3A = -59/72$$

6. $\cos(\theta - x) + \cos(\theta - y) + \cos(\theta - z)$

$$= \cos \theta \left(\sum \cos x \right) + \sin \theta \left(\sum \sin x \right) = 0$$

7. $\sin A \sin(B - C) = \sin C \sin(A - B)$

$$\Rightarrow \frac{\sin(B - C)}{\sin C \sin B} = \frac{\sin(A - B)}{\sin A \sin B}$$

$$\Rightarrow \frac{\sin B \cos C - \sin C \cos B}{\sin C \sin B} = \frac{\sin A \cos B - \sin B \cos A}{\sin A \sin B}$$

$$\Rightarrow \cot C - \cot B = \cot B - \cot A$$

$$\Rightarrow 2 \cot B = \cot A + \cot C$$

$$\therefore \cot A, \cot B, \cot C \text{ are in A.P.}$$

8. Multiplying and dividing each factor of numerator with $\sqrt{2}$, we get

$$E = \frac{\sqrt{2} \sin(45^\circ + 1^\circ) \cdot \sqrt{2} \sin(45^\circ + 2^\circ) \dots (\sqrt{2} \sin(45^\circ + 45^\circ))}{\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 45^\circ}$$

$$= (\sqrt{2})^{45} \left(\frac{\sin 46^\circ}{\cos 44^\circ} \right) \left(\frac{\sin 47^\circ}{\cos 43^\circ} \right) \left(\frac{\sin 48^\circ}{\cos 42^\circ} \right) \dots \left(\frac{\sin 89^\circ}{\cos 1^\circ} \right) \left(\frac{\sin 90^\circ}{\cos 45^\circ} \right)$$

$$= 2^{23}$$

9. $\sqrt{3} \sin x + \cos x = 2 \left(\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x \right) = 2 \sin \left(x + \frac{\pi}{6} \right)$

which is maximum when $x + \pi/6 = \pi/2$ or $x = 60^\circ$ and has a maximum value 2.

10. $1 + \sin \left(\frac{\pi}{4} + \theta \right) + 2 \sin \left(\frac{\pi}{4} - \theta \right)$

$$= 1 + \sin \left(\frac{\pi}{4} + \theta \right) + 2 \cos \left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \theta \right) \right)$$

$$= 1 + \sin \left(\frac{\pi}{4} + \theta \right) + 2 \cos \left(\frac{\pi}{4} + \theta \right)$$

Hence, the maximum value is $1 + \sqrt{5}$.

11. Since AM of two positive quantities $\geq$ their GM,

$$\frac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x} \cdot 2^{\cos x}}$$

$$= \sqrt{2^{\sin x + \cos x}}$$

$$= \sqrt{2^{\sqrt{2} \sin \left(x + \frac{\pi}{4} \right)}} \geq \sqrt{2^{-\sqrt{2}}}$$

$$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2.2 \cdot 2^{-\frac{1}{\sqrt{2}}} = 2^{1 - \frac{1}{\sqrt{2}}}$$

Exercise 3.2

1. $\frac{\cot A}{1 + \cot A} \times \frac{\cot B}{1 + \cot B}$

$$= \frac{1}{(1 + \tan A)(1 + \tan B)}$$

$$= \frac{1}{\tan A + \tan B + 1 + \tan A \tan B}$$

Now, $\tan(A + B) = \tan 225^\circ$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

Hence, from Eq. (i),

$$\text{R.H.S.} = \frac{1}{1 - \tan A \tan B + 1 + \tan A \tan B} = \frac{1}{2}$$

2. $\cot(A - B) = \frac{1}{\tan(A - B)} = \frac{1 + \tan A \tan B}{\tan A - \tan B}$

$$= \frac{1}{\tan A - \tan B} + \frac{\tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\cot B - \cot A}$$

$$= \frac{1}{x} + \frac{1}{y}$$

$$\begin{aligned} \frac{\tan^2 2\theta - \tan^2 \theta}{1 - \tan^2 2\theta \tan^2 \theta} &= \left(\frac{\tan 2\theta - \tan \theta}{1 + \tan 2\theta \tan \theta} \right) \left(\frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \right) \\ &= \tan(2\theta - \theta) \tan(2\theta + \theta) \\ &= \tan 3\theta \tan \theta \end{aligned}$$

$$4. \text{ Since } A + B = 45^\circ, \tan(A + B) = \tan 45^\circ = 1$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\text{or } \tan A + \tan B = 1 - \tan A \tan B$$

$$\text{or } \tan A + \tan A \tan B + \tan B = 1$$

$$\text{or } \tan A (1 + \tan B) + \tan B = 1$$

$$\text{or } \tan A (1 + \tan B) + 1 + \tan B = 1 + 1$$

$$\text{or } (1 + \tan A)(1 + \tan B) = 2$$

$$5. \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1$$

$$\Rightarrow A + B = 45^\circ \text{ (one of the possible values).}$$

$$\therefore 2A = 90^\circ - 2B$$

$$\Rightarrow \cos 2A = \sin 2B$$

For other values of $A + B$, we get the same result.

$$6. \text{ We have } P + Q = 210^\circ$$

$$\Rightarrow (\sqrt{3} + \tan P)(\sqrt{3} + \tan Q)$$

$$= (\sqrt{3} + \tan P)(\sqrt{3} + \tan(210^\circ - P))$$

$$= (\sqrt{3} + \tan P)(\sqrt{3} + \tan(30^\circ - P))$$

$$= (\sqrt{3} + \tan P) \left(\sqrt{3} + \frac{\left(\frac{1}{\sqrt{3}} - \tan P \right)}{\left(1 + \frac{1}{\sqrt{3}} \tan P \right)} \right)$$

$$= (\sqrt{3} + \tan P) \left(\sqrt{3} + \frac{1 - \sqrt{3} \cdot \tan P}{\sqrt{3} + \tan P} \right)$$

$$= 4$$

$$7. \tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha} = \frac{\left[\frac{n \sin \alpha \cos \alpha}{\cos^2 \alpha} \right]}{\left[\frac{1}{\cos^2 \alpha} - \frac{n \sin^2 \alpha}{\cos^2 \alpha} \right]}$$

[Dividing numerator and denominator by $\cos^2 \alpha$]

$$= \frac{n \tan \alpha}{\sec^2 \alpha - n \tan^2 \alpha} = \frac{n \tan \alpha}{1 + \tan^2 \alpha - n \tan^2 \alpha}$$

$$= \frac{n \tan \alpha}{1 + (1 - n) \tan^2 \alpha} \quad \dots(i)$$

$$\text{Now, } \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$= \frac{\left[\tan \alpha - \frac{n \tan \alpha}{1 + (1 - n) \tan^2 \alpha} \right]}{\left[1 + \tan \alpha \frac{n \tan \alpha}{1 + (1 - n) \tan^2 \alpha} \right]}$$

[From Eq. (i)]

$$= \frac{\tan \alpha + (1 - n) \tan^3 \alpha - n \tan \alpha}{1 + (1 - n) \tan^2 \alpha + n \tan^2 \alpha}$$

$$= \frac{(1 - n) \tan \alpha + (1 - n) \tan^3 \alpha}{1 + \tan^2 \alpha}$$

$$= (1 - n) \tan \alpha$$

Exercise 3.3

$$1. (a) \sin 65^\circ + \cos 65^\circ = \sin 65^\circ + \sin 25^\circ$$

$$= 2 \sin \left(\frac{65^\circ + 25^\circ}{2} \right) \cos \left(\frac{65^\circ - 25^\circ}{2} \right)$$

$$= 2 \sin 45^\circ \cos 20^\circ$$

$$= 2 \frac{1}{\sqrt{2}} \cos 20^\circ$$

$$= \sqrt{2} \cos 20^\circ$$

$$(b) \sin 47^\circ + \cos 77^\circ = \sin 47^\circ + \sin 13^\circ$$

$$= 2 \sin \left(\frac{47^\circ + 13^\circ}{2} \right) \cos \left(\frac{47^\circ - 13^\circ}{2} \right)$$

$$= 2 \sin 30^\circ \cos 17^\circ$$

$$= 2 \left(\frac{1}{2} \right) \cos 17^\circ$$

$$= \cos 17^\circ$$

$$2. \cos 80^\circ + \cos 40^\circ - \cos 20^\circ$$

$$= 2 \cos \left(\frac{80^\circ + 40^\circ}{2} \right) \cos \left(\frac{80^\circ - 40^\circ}{2} \right) - \cos 20^\circ$$

$$= 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ$$

$$= 2 \frac{1}{2} \cos 20^\circ - \cos 20^\circ = 0$$

$$3. \sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ$$

$$= 2 \sin \left(\frac{10^\circ + 50^\circ}{2} \right) \cos \left(\frac{50^\circ - 10^\circ}{2} \right)$$

$$+ 2 \sin \left(\frac{20^\circ + 40^\circ}{2} \right) \cos \left(\frac{40^\circ - 20^\circ}{2} \right)$$

$$= 2 \sin 30^\circ \cos 20^\circ + 2 \sin 30^\circ \cos 10^\circ$$

$$= 2 \frac{1}{2} \sin 70^\circ + 2 \frac{1}{2} \sin 80^\circ$$

$$= \sin 70^\circ + \sin 80^\circ$$

$$4. \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} + \cos \frac{6\pi}{5} + \cos \frac{7\pi}{5}$$

$$= \left(\cos \frac{\pi}{5} + \cos \frac{7\pi}{5} \right) + \left(\cos \frac{2\pi}{5} + \cos \frac{6\pi}{5} \right)$$

$$= 2 \cos \frac{4\pi}{5} \cos \frac{3\pi}{5} + 2 \cos \frac{4\pi}{5} \cos \frac{2\pi}{5}$$

$$= 2 \cos \frac{4\pi}{5} \left(\cos \frac{3\pi}{5} + \cos \frac{2\pi}{5} \right)$$

$$= 2 \cos \frac{4\pi}{5} \left(2 \cos \frac{\pi}{2} \cos \frac{\pi}{10} \right) = 0$$

$$5. \sin \alpha - \sin \beta = \frac{1}{3}$$

$$\Rightarrow 2 \sin \left(\frac{\alpha - \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right) = \frac{1}{3} \quad \dots(i)$$

$$\cos \beta - \cos \alpha = \frac{1}{2}$$

$$\Rightarrow 2 \sin \left(\frac{\alpha - \beta}{2} \right) \sin \left(\frac{\alpha + \beta}{2} \right) = \frac{1}{2} \quad \dots(ii)$$

Dividing Eqs. (i) by (ii), we have $\cot \frac{\alpha + \beta}{2} = \frac{2}{3}$

6. $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$

or $\frac{1}{\sin A} - \frac{1}{\sin B} = \frac{1}{\cos B} - \frac{1}{\cos A}$

or $\frac{\sin B - \sin A}{\sin A \sin B} = \frac{\cos A - \cos B}{\cos A \cos B}$

or $\frac{2 \sin \left(\frac{B-A}{2} \right) \cos \left(\frac{A+B}{2} \right)}{\sin A \sin B} = \frac{2 \sin \left(\frac{B-A}{2} \right) \sin \left(\frac{A+B}{2} \right)}{\cos A \cos B}$

or $\frac{\cos \left(\frac{A+B}{2} \right)}{\sin \left(\frac{A+B}{2} \right)} = \frac{\sin A \sin B}{\cos A \cos B}$

or $\tan A \tan B = \cot \left(\frac{A+B}{2} \right)$

7. $\sin 25^\circ \cos 115^\circ$

$$\begin{aligned} &= \frac{1}{2} (2 \sin 25^\circ \cos 115^\circ) \\ &= \frac{1}{2} [\sin(25^\circ + 115^\circ) + \sin(25^\circ - 115^\circ)] \\ &= \frac{1}{2} (\sin 140^\circ - \sin 90^\circ) \\ &= \frac{1}{2} (\sin(180^\circ - 40^\circ) - 1) \\ &= \frac{1}{2} (\sin 40^\circ - 1) \end{aligned}$$

8. Let

$$x \cos \theta = y \cos \left(\theta + \frac{2\pi}{3} \right) = z \cos \left(\theta + \frac{4\pi}{3} \right) = k \quad \dots(i)$$

Now, $xy + yz + zx = xyz \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$

$$= \frac{xyz}{k} \left(\cos \theta + \left\{ \cos \left(\theta + \frac{2\pi}{3} \right) + \cos \left(\theta + \frac{4\pi}{3} \right) \right\} \right)$$

$$= \frac{xyz}{k} \left(\cos \theta + 2 \cos \left(\frac{\pi}{3} \right) \cos(\theta + \pi) \right)$$

$$= \frac{xyz}{k} \left(\cos \theta - 2 \frac{1}{2} \cos \theta \right) = 0$$

9. $y \sin \phi = x \sin(2\theta + \phi)$

or $\frac{x}{y} = \frac{\sin \phi}{\sin(2\theta + \phi)}$

or $\frac{x+y}{y-x} = \frac{\sin \phi + \sin(2\theta + \phi)}{\sin(2\theta + \phi) - \sin \phi}$

[applying componendo and dividendo]

$$= \frac{2 \sin(\theta + \phi) \cos \theta}{2 \sin \theta \cos(\theta + \phi)}$$

$$\Rightarrow (x+y) \cot(\theta + \phi) = (y-x) \cot \theta$$

10. $\cos(A+B) \sin(C+D) = \cos(A-B) \sin(C-D)$

or $\frac{\cos(A+B)}{\cos(A-B)} = \frac{\sin(C-D)}{\sin(C+D)}$

or $\frac{\cos(A+B) + \cos(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{\sin(C-D) + \sin(C+D)}{\sin(C-D) - \sin(C+D)}$

or $\frac{2 \cos A \cos B}{-2 \sin A \sin B} = \frac{2 \sin C \cos D}{-2 \sin D \cos C}$

or $\cot A \cot B = \tan C \cot D$

or $\cot A \cot B \cot C = \cot D$

11. (a) $\tan(A+B) = 3 \tan A$

or $\frac{\sin(A+B) \cos A}{\sin A \cos(A+B)} = 3$

or $\frac{\sin(A+B) \cos A + \sin A \cos(A+B)}{\sin(A+B) \cos A - \sin A \cos(A+B)} = \frac{3+1}{3-1}$

or $\frac{\sin(A+B+A)}{\sin(A+B-A)} = 2$

or $\sin(2A+B) = 2 \sin B$

(b) $\sin 2(A+B) + \sin 2A$
 $= 2 \sin(2A+B) \cos B$
 $= 4 \sin B \cos B$

12. $\frac{x \tan A + y \tan B}{x+y} = \frac{\frac{x}{y} \tan A + \tan B}{\frac{x}{y} + 1}$
 $= \frac{\sin A + \sin B}{\cos A + \cos B}$
 $= \tan \frac{A+B}{2}$

13. $\frac{\cos 6x + 6 \cos 4x + 15 \cos 2x + 10}{\cos 5x + 5 \cos 3x + 10 \cos x} = 1$

$$\Rightarrow \frac{(\cos 6x + \cos 4x) + 5(\cos 4x + \cos 2x) + 10(\cos 2x + 1)}{\cos 5x + 5 \cos 3x + 10 \cos x} = 1$$

$$\Rightarrow \frac{2 \cos 5x \cos x + 2 \times 5 \cos 3x \cos x + 10 \times 2 \cos^2 x}{\cos 5x + 5 \cos 3x + 10 \cos x} = 1$$

$$\Rightarrow \frac{2 \cos x [\cos 5x + 5 \cos 3x + 10 \cos x]}{\cos 5x + 5 \cos 3x + 10 \cos x} = 1$$

$$\Rightarrow \cos x = \frac{1}{2}$$

$$\therefore x = 60^\circ$$

Exercise 3.4

1. $\frac{1 + \sin 2A - \cos 2A}{1 + \sin 2A + \cos 2A} = \frac{(1 - \cos 2A) + \sin 2A}{(1 + \cos 2A) + \sin 2A}$
 $= \frac{2 \sin^2 A + 2 \sin A \cos A}{2 \cos^2 A + 2 \sin A \cos A}$
 $= \frac{2 \sin A (\sin A + \cos A)}{2 \cos A (\sin A + \cos A)} = \tan A$

2. $\frac{1 + \sin 2A}{\cos 2A} = \frac{\sin^2 A + \cos^2 A + 2 \sin A \cos A}{\cos^2 A - \sin^2 A}$
 $= \frac{(\sin A + \cos A)^2}{(\cos A - \sin A)(\sin A + \cos A)}$
 $= \frac{\cos A + \sin A}{\cos A - \sin A}$
 $= \frac{1 + \tan A}{1 - \tan A} = \tan \left(\frac{\pi}{4} + A \right)$

$$\begin{aligned}
 3. \cot \theta - \tan \theta &= \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \\
 &= \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} \\
 &= \frac{2 \cos 2\theta}{\sin 2\theta} \\
 &= 2 \cot 2\theta
 \end{aligned}$$

$$\begin{aligned}
 4. \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} &= \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \cdot \frac{\cos \theta - \sin \theta}{\cos \theta - \sin \theta} \\
 &= \frac{\cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta}{\cos^2 \theta - \sin^2 \theta} \\
 &= \frac{1 - \sin 2\theta}{\cos 2\theta} \\
 &= \sec 2\theta - \tan 2\theta
 \end{aligned}$$

$$\begin{aligned}
 5. \tan\left(\frac{\pi}{4} + \theta\right) - \tan\left(\frac{\pi}{4} - \theta\right) &= \frac{1 + \tan \theta}{1 - \tan \theta} - \frac{1 - \tan \theta}{1 + \tan \theta} \\
 &= \frac{(1 + \tan \theta)^2 - (1 - \tan \theta)^2}{1 - \tan^2 \theta} \\
 &= \frac{4 \tan \theta}{1 - \tan^2 \theta} \\
 &= 2 \tan 2\theta
 \end{aligned}$$

$$\begin{aligned}
 6. \operatorname{cosec} A - 2 \cot 2A \cos A &= \frac{1}{\sin A} - \frac{2 \cos 2A \cos A}{\sin 2A} \\
 &= \frac{1}{\sin A} - \frac{2 \cos 2A \cos A}{2 \sin A \cos A} \\
 &= \frac{1}{\sin A} - \frac{\cos 2A}{\sin A} \\
 &= \frac{1 - \cos 2A}{\sin A} \\
 &= \frac{2 \sin^2 A}{\sin A} \\
 &= 2 \sin A
 \end{aligned}$$

$$\begin{aligned}
 7. \cos^3 \theta \times \sin 3\theta + \sin^3 \theta \times \cos 3\theta &= \cos^3 \theta \times (3 \sin \theta - 4 \sin^3 \theta) + \sin^3 \theta (4 \cos^3 \theta - 3 \cos \theta) \\
 &= 3 \cos^3 \theta \sin \theta - 4 \sin^3 \theta \cos^3 \theta + 4 \sin^3 \theta \cos^3 \theta - 3 \sin^3 \theta \cos \theta \\
 &= 3 \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta) \\
 &= \frac{3}{2} \sin 2\theta \cos 2\theta \\
 &= \frac{3}{4} \sin 4\theta
 \end{aligned}$$

$$\begin{aligned}
 8. \frac{\sin^2 3A}{\sin^2 A} - \frac{\cos^2 3A}{\cos^2 A} &= \frac{\sin^2 3A \cos^2 A - \cos^2 3A \sin^2 A}{\sin^2 A \cos^2 A} \\
 &= \frac{\sin^2 3A (1 - \sin^2 A) - \cos^2 3A \sin^2 A}{\sin^2 A \cos^2 A} \\
 &= \frac{\sin^2 3A - \sin^2 A (\cos^2 3A + \sin^2 3A)}{\sin^2 A \cos^2 A} \\
 &= \frac{\sin(3A + A) \sin(3A - A)}{\sin^2 A \cos^2 A}
 \end{aligned}$$

$$= \frac{(4 \sin A \cos A \cos 2A)(2 \sin A \cos A)}{\sin^2 A \cos^2 A} = 8 \cos 2A$$

$$\begin{aligned}
 9. \text{R.H.S.} &= \frac{1 + \cos 2\theta}{\cos 2\theta} \times \frac{1 + \cos 4\theta}{\cos 4\theta} \times \frac{1 + \cos 8\theta}{\cos 8\theta} \\
 &= \frac{2 \cos^2 \theta \times 2 \cos^2 2\theta \times 2 \cos^2 4\theta}{\cos 2\theta \cos 4\theta \cos 8\theta} \\
 &= \frac{[8 \cos \theta \cos 2\theta \cos 4\theta] \cos \theta}{\cos 8\theta} = \frac{\left[\frac{\sin 8\theta}{\sin \theta} \right] \cos \theta}{\cos 8\theta} = \frac{\tan 8\theta}{\tan \theta}
 \end{aligned}$$

$$10. \text{We have } \sin 10^\circ = \frac{a}{2b}$$

$$\text{Now } \sin 30^\circ = 3 \sin 10^\circ - 4 \sin^3 10^\circ$$

$$\Rightarrow \frac{1}{2} = \frac{3a}{2b} - \frac{4a^3}{8b^3}$$

$$\Rightarrow 1 = \frac{3a}{b} - \frac{a^3}{b^3}$$

$$\Rightarrow a^3 + b^3 = 3ab^2$$

$$11. E = \sin A + \sin 2B + \sin 3C$$

$$= \frac{3}{5} + 2 \times \frac{4}{5} \times \frac{3}{5} - 1$$

$$= \frac{15}{25} + \frac{24}{25} - 1 = \frac{39 - 25}{25} = \frac{14}{25}$$

$$12. 32 \sin\left(\frac{A}{2}\right) \sin\left(\frac{5A}{2}\right)$$

$$= 16 \left(\cos\left(\frac{A}{2} - \frac{5A}{2}\right) - \cos\left(\frac{A}{2} + \frac{5A}{2}\right) \right)$$

$$= 16 (\cos 2A - \cos 3A)$$

$$= 16 (2 \cos^2 A - 1 - 4 \cos^3 A + 3 \cos A)$$

Now, putting the value of $\cos A = 3/4$, we get

$$32 \sin\left(\frac{A}{2}\right) \sin\left(\frac{5A}{2}\right) = 11$$

$$13. \text{We have } 4 \cos^2 \theta - 1 = 3 - 4 \sin^2 \theta = \frac{\sin 3\theta}{\sin \theta}$$

$\therefore$ Given expression

$$= (4 \cos^2 \theta - 1)(4 \cos^2 3\theta - 1)(4 \cos^2 9\theta - 1)(4 \cos^2 27\theta - 1),$$

where $\theta = 9^\circ$

$$= \frac{\sin 3\theta}{\sin \theta} \cdot \frac{\sin 9\theta}{\sin 3\theta} \cdot \frac{\sin 27\theta}{\sin 9\theta} \cdot \frac{\sin 81\theta}{\sin 27\theta}$$

$$= \frac{\sin 81\theta}{\sin \theta} = \frac{\sin 729^\circ}{\sin 9^\circ} = 1$$

$$14. \cos^2 \frac{\theta}{2} = 1 - \sin^2 \frac{\theta}{2} = 1 - \frac{x-1}{2x} = \frac{x+1}{2x}$$

$$\Rightarrow \tan \frac{\theta}{2} = \sqrt{\frac{x-1}{2x}} + \sqrt{\frac{x+1}{2x}} = \sqrt{\frac{x-1}{x+1}}$$

$$\Rightarrow \tan \theta = \frac{2 \tan(\theta/2)}{1 - \tan^2(\theta/2)}$$

$$= \frac{2 \sqrt{\frac{x-1}{x+1}}}{1 - \frac{x-1}{x+1}} = \frac{2 \sqrt{x-1} \sqrt{x+1}}{1 - \frac{x-1}{x+1}} = \sqrt{x-1} \sqrt{x+1} = \sqrt{x^2 - 1}$$



$$15. \sin A \sin (B - C) = \sin C \sin (A - B)$$

$$\Rightarrow \sin (B + C) \sin (B - C) = \sin (A + B) \sin (A - B)$$

$$\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$

$$\Rightarrow 2 \sin^2 B = \sin^2 A + \sin^2 C$$

$$\Rightarrow 1 - \cos 2B = \frac{1}{2}(1 - \cos 2A) + \frac{1}{2}(1 - \cos 2C),$$

$$\Rightarrow 2 \cos 2B = \cos 2A + \cos 2C$$

Thus, $\cos 2A, \cos 2B, \cos 2C$ are in A.P.

$$16. (a) \sin^2 3a - \sin^2 a = \sin (3a + a) \sin (3a - a) \\ = \sin 4a \sin 2a$$

$$\text{Now, } \sin 4a = \sin \frac{4\pi}{7} \\ = \sin \left(\pi - \frac{4\pi}{7} \right) \\ = \sin \frac{3\pi}{7} \\ = \sin 3a$$

$$\therefore \sin^2 3a - \sin^2 a = \sin 3a \sin 2a$$

$$(b) \operatorname{cosec} 2a + \operatorname{cosec} 4a = \frac{\sin 4a + \sin 2a}{\sin 4a \sin 2a} \\ = \frac{2 \sin 3a \cos a}{\sin 4a \sin 2a} \\ = \frac{2 \sin 3a \cos a}{\sin 4a (2 \sin a \cos a)} \\ = \frac{1}{\sin a} \\ = \operatorname{cosec} a$$

$$(c) a = \frac{\pi}{7}$$

$$\therefore 7a = 3a + 4a = \pi$$

$$\therefore \sin 3a = \sin 4a$$

$$\therefore \sin a (3 - 4 \sin^2 a) = 2 \sin 2a \cos 2a \\ = 4 \sin a \cos a \cos 2a,$$

$$\therefore 3 - 4(1 - \cos^2 a) = 4 \cos a (2 \cos^2 a - 1) \quad (\because \sin a \neq 0)$$

It follows that

$$8 \cos^3 a - 4 \cos^2 a - 4 \cos a + 1 = 0 \quad \dots(i)$$

From equation (i), we can say that $\cos a$ is a root of

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

$$17. \frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = \frac{\cos 10^\circ - \sqrt{3} \sin 10^\circ}{\sin 10^\circ \cos 10^\circ} \\ = 4 \times \frac{\frac{1}{2} \cos 10^\circ - \frac{\sqrt{3}}{2} \sin 10^\circ}{2 \sin 10^\circ \cos 10^\circ} \\ = 4 \times \frac{\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ}{\sin 20^\circ} \\ = 4 \times \frac{\sin 20^\circ}{\sin 20^\circ} = 4$$

$$18. 2 \sin^2 \beta + 4 \cos(\alpha + \beta) \sin \alpha \sin \beta + \cos 2(\alpha + \beta) \\ = 2 \sin^2 \beta + 2 \cos(\alpha + \beta) [2 \sin \alpha \sin \beta] + \cos 2(\alpha + \beta) \\ = 2 \sin^2 \beta + 2 \cos(\alpha + \beta) [\cos(\alpha - \beta) - \cos(\alpha + \beta)] + \cos 2(\alpha + \beta) \\ = 2 \sin^2 \beta + 2 \cos(\alpha + \beta) \cos(\alpha - \beta) - 2 \cos^2(\alpha + \beta) \\ + 2 \cos^2(\alpha + \beta) - 1$$

$$= 2 \sin^2 \beta + 2 (\cos^2 \alpha - \sin^2 \beta) - 1$$

$$= 2 \cos^2 \alpha - 1$$

$$= \cos 2\alpha$$

$$19. \tan 2x = \frac{b}{a+b} \\ \Rightarrow \frac{2 \tan x}{1 - \tan^2 x} = \frac{b}{a+b}$$

$$\Rightarrow \frac{\frac{2a}{b}}{1 - \frac{a^2}{b^2}} = \frac{b}{a+b}$$

$$\Rightarrow \frac{2ab}{b-a} = b$$

$$\Rightarrow 2a = b - a$$

$$\Rightarrow 3a = b$$

$$\Rightarrow \frac{a}{b} = \frac{1}{3}$$

$$\text{Now, } \tan x = \frac{a}{b}$$

$$\Rightarrow x = \tan^{-1} \frac{1}{3}$$

$$20. \tan \theta + \tan(60^\circ + \theta) + \tan(120^\circ + \theta) \\ = \tan \theta + \frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \tan \theta} + \frac{-\sqrt{3} + \tan \theta}{1 + \sqrt{3} \tan \theta} \\ = \frac{[\tan \theta(1 - 3 \tan^2 \theta) + (\sqrt{3} + \tan \theta)(1 + \sqrt{3} \tan \theta) \\ + (-\sqrt{3} + \tan \theta)(1 - \sqrt{3} \tan \theta)]}{1 - 3 \tan^2 \theta} \\ = \frac{[\tan \theta - 3 \tan^3 \theta + \sqrt{3} + 4 \tan \theta + \sqrt{3} \tan^2 \theta \\ - \sqrt{3} - \sqrt{3} \tan^2 \theta + 4 \tan \theta]}{1 - 3 \tan^2 \theta} \\ = \frac{9 \tan \theta - 3 \tan^3 \theta}{1 - 3 \tan^2 \theta} = 3 \tan 3\theta$$

$$21. \frac{1 + \sqrt{1 + \tan^2 2A}}{\tan 2A} = \frac{1 + |\sec 2A|}{\tan 2A} \quad (2A = 225^\circ) \\ = \frac{1 - \sec 2A}{\tan 2A} = - \left(\frac{1 - \cos 2A}{\sin 2A} \right) = -\tan A$$

22. Since α and β are the roots of the equation

$$a \cos \theta + b \sin \theta = c, \text{ we have}$$

$$a \cos \alpha + b \sin \alpha = c$$

$$\text{and } a \cos \beta + b \sin \beta = c$$

Subtracting Eq. (ii) from Eq. (i), we get

$$a(\cos \alpha - \cos \beta) + b(\sin \alpha - \sin \beta) = 0$$

$$\text{or } b(\sin \alpha - \sin \beta) - a(\cos \beta - \cos \alpha) = 0$$

$$\text{or } 2b \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} = 2a \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\text{or } \tan \frac{\alpha + \beta}{2} = \frac{b}{a} \quad \left[\text{as } \alpha, \beta \text{ are different, } \sin \frac{\alpha - \beta}{2} \neq 0 \right]$$

$$\text{Now, } \sin(\alpha + \beta) = \frac{2 \tan \frac{\alpha + \beta}{2}}{1 + \tan^2 \frac{\alpha + \beta}{2}} = \frac{2 \frac{b}{a}}{1 + \frac{b^2}{a^2}} = \frac{2ab}{a^2 + b^2}$$

$$(a) \tan(\alpha + \beta) = \frac{2 \tan\left(\frac{\alpha + \beta}{2}\right)}{1 - \tan^2\left(\frac{\alpha + \beta}{2}\right)} = \frac{2ab}{a^2 - b^2}$$

$$(b) \cos(\alpha + \beta) = \frac{1 - \tan^2\left(\frac{\alpha + \beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha + \beta}{2}\right)} = \frac{1 - \frac{b^2}{a^2}}{1 + \frac{b^2}{a^2}} = \frac{a^2 - b^2}{a^2 + b^2}$$

$$23. \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\tan \alpha - \tan \beta}{\tan \alpha + \tan \beta} = \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)}$$

$$\begin{aligned} 24. \tan^2 \frac{\theta}{2} + \tan^2 \frac{\phi}{2} + \tan^2 \frac{\psi}{2} &= \frac{1 - \cos \theta}{1 + \cos \theta} + \frac{1 - \cos \phi}{1 + \cos \phi} + \frac{1 - \cos \psi}{1 + \cos \psi} \\ &= \frac{1 - \frac{a}{b+c}}{1 + \frac{a}{b+c}} + \frac{1 - \frac{b}{c+a}}{1 + \frac{b}{c+a}} + \frac{1 - \frac{c}{a+b}}{1 + \frac{c}{a+b}} \\ &= \frac{b+c-a}{a+b+c} + \frac{c+a-b}{a+b+c} + \frac{a+b-c}{a+b+c} \\ &= \frac{a+b+c}{a+b+c} = 1 \end{aligned}$$

$$25. \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta}$$

$$= \frac{1 - \frac{\cos \alpha - \cos \beta}{1 - \cos \alpha \cos \beta}}{1 + \frac{\cos \alpha - \cos \beta}{1 - \cos \alpha \cos \beta}}$$

$$= \frac{1 - \cos \alpha \cos \beta - \cos \alpha + \cos \beta}{1 - \cos \alpha \cos \beta + \cos \alpha - \cos \beta}$$

$$= \frac{(1 - \cos \alpha) + \cos \beta (1 - \cos \alpha)}{(1 + \cos \alpha) - \cos \beta (1 + \cos \alpha)}$$

$$= \frac{(1 - \cos \alpha)(1 + \cos \beta)}{(1 + \cos \alpha)(1 - \cos \beta)} = \tan^2 \frac{\alpha}{2} \cot^2 \frac{\beta}{2}$$

$$\therefore \tan \frac{\theta}{2} = \pm \tan \frac{\alpha}{2} \cot \frac{\beta}{2}$$

Hence, one of the values of $\tan \frac{\theta}{2}$ is $\tan \frac{\alpha}{2} \cot \frac{\beta}{2}$.

26. Let us put $\tan \theta = t_1$ and $\tan \phi = t_2$. Then,

$$t_1^2 t_2^2 = \frac{a-b}{a+b} \quad \dots(i)$$

$$\text{Also, } \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - t_1^2}{1 + t_1^2}$$

$$\cos 2\phi = \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} = \frac{1 - t_2^2}{1 + t_2^2}$$

$$\begin{aligned} \text{Now, } a - b \cos 2\theta &= a - b \frac{(1 - t_1^2)}{(1 + t_1^2)} \\ &= \frac{(a - b) + (a + b)t_1^2}{1 + t_1^2} \\ &= \frac{(a + b)}{(1 + t_1^2)} \left[\frac{a - b}{a + b} + t_1^2 \right] \end{aligned}$$

$$= \frac{a + b}{1 + t_1^2} [t_1^2 t_2^2 + t_1^2] \quad [\text{using Eq. (i)}]$$

$$= \frac{a + b}{(1 + t_1^2)} (t_2^2 + 1) t_1^2 \quad \dots(ii)$$

$$\text{Similarly, } a - b \cos 2\phi = \frac{a + b}{(1 + t_2^2)} t_2^2 (t_1^2 + 1) \quad \dots(iii)$$

$$\begin{aligned} \text{Hence, } (a - b \cos 2\theta)(a - b \cos 2\phi) &= (a + b)^2 t_1^2 t_2^2 = a^2 - b^2 \end{aligned}$$

[using Eq. (i)]

which is independent if θ and ϕ .

Exercise 3.5

$$\begin{aligned} 1. (\cos^2 66^\circ - \sin^2 6^\circ)(\cos^2 48^\circ - \sin^2 12^\circ) &= (\cos 72^\circ \cdot \cos 60^\circ)(\cos 60^\circ \cdot \cos 36^\circ) \\ &= (\cos 60^\circ)^2 \cdot \sin 18^\circ \cdot \cos 36^\circ \\ &= \frac{1}{4} \left(\frac{\sqrt{5} - 1}{4} \right) \left(\frac{\sqrt{5} + 1}{4} \right) \\ &= \frac{1}{16} \end{aligned}$$

$$\begin{aligned} 2. 4(\sin 24^\circ + \cos 6^\circ) &= 4(\sin 24^\circ + \sin 84^\circ) \\ &= 8 \sin 54^\circ \cos 30^\circ \\ &= 8 \times \frac{\sqrt{5} + 1}{4} \times \frac{\sqrt{3}}{2} \\ &= \sqrt{15} + \sqrt{3} \end{aligned}$$

$$\begin{aligned} 3. \sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ &= (\sin 47^\circ - \sin 25^\circ) + (\sin 61^\circ - \sin 11^\circ) \\ &= 2 \sin 11^\circ \cos 36^\circ + 2 \sin 25^\circ \cos 36^\circ \\ &= 2 \cos 36^\circ (\sin 11^\circ + \sin 25^\circ) \\ &= 2 \cos 36^\circ (2 \sin 18^\circ \cos 7^\circ) \\ &= 4 \times \frac{\sqrt{5} + 1}{4} \times \frac{\sqrt{5} - 1}{4} \times \cos 7^\circ \\ &= \cos 7^\circ \end{aligned}$$

$$\begin{aligned} 4. (a) \frac{1 + \tan^2 7\frac{1}{2}^\circ}{\tan 7\frac{1}{2}^\circ} &= 2 \frac{1 + \tan^2 7\frac{1}{2}^\circ}{2 \tan 7\frac{1}{2}^\circ} \\ &= \frac{2}{\sin 15^\circ} \\ &= \frac{2}{\sqrt{3} - 1} \\ &= 2\sqrt{2}(\sqrt{3} + 1) \\ &= 2(\sqrt{6} + \sqrt{2}) \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \frac{\tan^2 82\frac{1}{2}^\circ - 1}{\tan 82\frac{1}{2}^\circ} &= -2 \frac{1 - \tan^2 82\frac{1}{2}^\circ}{2 \tan 82\frac{1}{2}^\circ} \\
 &= \frac{-2}{\tan 165^\circ} \\
 &= \frac{2}{\tan 15^\circ} \\
 &= \frac{2}{2 - \sqrt{3}} \\
 &= 2(2 + \sqrt{3})
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{\tan^2 37\frac{1}{2}^\circ + 1}{\tan^2 37\frac{1}{2}^\circ - 1} &= \frac{-1}{\cos 75^\circ} \\
 &= \frac{-2\sqrt{2}}{\sqrt{3} - 1} \\
 &= -\sqrt{2}(\sqrt{3} + 1) \\
 &= -(\sqrt{6} + \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \frac{3 \tan^2 5^\circ - 1}{3 \tan 5^\circ - \tan^3 5^\circ} &= \frac{-1}{\tan 15^\circ} \\
 &= \frac{-1}{2 - \sqrt{3}} \\
 &= -(2 + \sqrt{3})
 \end{aligned}$$

$$\begin{aligned}
 5. \quad &\cot\left(11\frac{1}{4}^\circ\right) + \tan\left(112\frac{1}{2}^\circ\right) - \cot\left(112\frac{1}{2}^\circ\right) - \tan\left(11\frac{1}{4}^\circ\right) \\
 &= \left(\cot 11\frac{1}{4}^\circ - \tan 11\frac{1}{4}^\circ\right) + \tan\left(90^\circ + 22\frac{1}{2}^\circ\right) - \cot\left(90^\circ + 22\frac{1}{2}^\circ\right) \\
 &= \frac{2\left(1 - \tan^2 11\frac{1}{4}^\circ\right)}{2 \tan 11\frac{1}{4}^\circ} - \cot 22\frac{1}{2}^\circ + \tan 22\frac{1}{2}^\circ \\
 &= 2 \cot 22\frac{1}{2}^\circ - \cot 22\frac{1}{2}^\circ + \tan 22\frac{1}{2}^\circ \\
 &= \cot 22\frac{1}{2}^\circ + \tan 22\frac{1}{2}^\circ \\
 &= (\sqrt{2} + 1) + (\sqrt{2} - 1) \\
 &= 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \frac{\tan 9^\circ + \cot 9^\circ}{\tan 27^\circ + \cot 27^\circ} &= \frac{\tan 9^\circ + \frac{1}{\tan 9^\circ}}{\tan 27^\circ + \frac{1}{\tan 27^\circ}} \\
 &= \frac{\tan^2 9^\circ + 1}{2 \tan 9^\circ} \cdot \frac{2 \tan 27^\circ}{(\tan^2 27^\circ + 1)} \\
 &= \frac{\sin 54^\circ}{\sin 18^\circ} = \frac{\cos 36^\circ}{\sin 18^\circ} \\
 &= \frac{\sqrt{5} + 1}{4} \cdot \frac{4}{\sqrt{5} - 1} \\
 &= \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \times \frac{\sqrt{5} + 1}{\sqrt{5} + 1} = \frac{3 + \sqrt{5}}{2}
 \end{aligned}$$

Exercise 3.6

$$\begin{aligned}
 1. \quad &\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ \\
 &= \sin 60^\circ \sin 20^\circ \sin (60^\circ - 20^\circ) \sin (60^\circ + 20^\circ) \\
 &= \frac{1}{4} \sin 60^\circ \sin (3 \times 20^\circ) = \frac{1}{4} \sin^2 60^\circ = \frac{3}{16}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad &\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ \\
 &= \cos 30^\circ \cos 10^\circ \cos (60^\circ - 10^\circ) \cos (60^\circ + 10^\circ) \\
 &= \frac{1}{4} \cos 30^\circ \cos (3 \times 10^\circ) = \frac{3}{16}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad &\sin 12^\circ \sin 18^\circ \sin 42^\circ \sin 48^\circ \sin 72^\circ \sin 78^\circ \\
 &= (\sin 12^\circ \sin 48^\circ \sin 72^\circ) (\sin 18^\circ \sin 42^\circ \sin 78^\circ) \\
 &= \frac{1}{4} \sin 36^\circ \frac{1}{4} \sin 54^\circ \\
 &= \frac{1}{16} \sin 36^\circ \cos 36^\circ \\
 &= \frac{1}{32} \sin 72^\circ \\
 &= \frac{\cos 18^\circ}{32}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad &\sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14} \\
 &= \sin \frac{5\pi}{14} \sin \frac{3\pi}{14} \sin \frac{\pi}{14} \\
 &= \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} \\
 &= -\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \\
 &= -\frac{\sin \frac{8\pi}{7}}{8 \sin \frac{\pi}{7}} = \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad &\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14} \\
 &= \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{\pi}{2} \sin \left(\pi - \frac{5\pi}{14}\right) \\
 &\quad \times \sin \left(\pi - \frac{3\pi}{14}\right) \sin \left(\pi - \frac{\pi}{14}\right) \\
 &= \sin^2 \frac{\pi}{14} \sin^2 \frac{3\pi}{14} \sin^2 \frac{5\pi}{14} \\
 &= \left(\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}\right)^2 \\
 &= \left[\cos \left(\frac{\pi}{2} - \frac{\pi}{14}\right) \cos \left(\frac{\pi}{2} - \frac{3\pi}{14}\right) \cos \left(\frac{\pi}{2} - \frac{5\pi}{14}\right)\right]^2 \\
 &= \left[\cos \frac{3\pi}{7} \cos \frac{2\pi}{7} \cos \frac{\pi}{7}\right]^2 \\
 &= \left[\frac{1}{2 \sin \pi/7} \left\{2 \cos \frac{\pi}{7} \sin \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7}\right\}\right]^2 \\
 &= \left[\frac{1}{2^2 \sin \pi/7} \left\{2 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \cos \left(\pi - \frac{4\pi}{7}\right)\right\}\right]^2 \\
 &= \left[\frac{1}{2^3 \sin \pi/7} \left(2 \sin \frac{4\pi}{7} \cos \frac{4\pi}{7}\right)\right]^2
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{1}{8 \sin \pi/7} \sin \frac{8\pi}{7} \right)^2 \\
 &= \left(\frac{\sin(\pi + \pi/7)}{8 \sin \pi/7} \right)^2 \\
 &= \left(\frac{-\sin \pi/7}{8 \sin \pi/7} \right)^2 = \left(\frac{1}{8} \right)^2 = \frac{1}{64}
 \end{aligned}$$

Exercise 3.7

1. We have

$$\begin{aligned}
 &\cos \frac{\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{7\pi}{11} + \cos \frac{9\pi}{11} \\
 &\quad \cos \left(\frac{5\pi}{11} \right) \cdot \sin \left(\frac{\frac{\pi}{11} + \frac{9\pi}{11}}{2} \right) \\
 &= \frac{\cos \left(\frac{5\pi}{11} \right) \cdot \sin \left(\frac{\pi}{2} \right)}{\sin \left(\frac{\pi}{11} \right)} \\
 &= \frac{\cos \frac{5\pi}{11} \sin \frac{5\pi}{11}}{\sin \frac{\pi}{11}} = \frac{1}{2} \frac{\sin \left(\frac{10\pi}{11} \right)}{\sin \frac{\pi}{11}} = \frac{1}{2}
 \end{aligned}$$

2. There are 90 terms and the average A is

$$\begin{aligned}
 A &= \frac{1}{90} \sum_{r=1}^{90} \sin 2r \\
 A &= \frac{1}{90} \times \frac{\sin \left(90^\circ \times \frac{2}{2} \right)}{\sin 1^\circ} \times \sin \left[\frac{2^\circ + 180^\circ}{2} \right] \\
 &= \frac{1}{90} \times \frac{1}{\sin 1^\circ} \times \sin(90^\circ + 1^\circ) \\
 &= \frac{1}{90} \times \frac{\cos 1^\circ}{\sin 1^\circ} = \frac{1}{90} (\cot 1^\circ)
 \end{aligned}$$

$$\begin{aligned}
 3. \sum_{r=1}^{n-1} \sin^2 \frac{r\pi}{n} &= \sum_{r=1}^{n-1} \frac{1 - \cos(2r\pi/n)}{2} \\
 &= \frac{1}{2} \left(n-1 - \frac{\sin(n-1) \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cos \left(\frac{2\pi}{n} + (n-1) \frac{2\pi}{n} \right) \right) \\
 &= \frac{1}{2} \left(n-1 - \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} (-1) \right) = \frac{n}{2}
 \end{aligned}$$

$$\begin{aligned}
 4. S &= \sqrt{1 + \cos \alpha} + \sqrt{1 + \cos 2\alpha} + \sqrt{1 + \cos 3\alpha} + \dots n \text{ terms} \\
 &= \sqrt{2 \cos^2 \frac{\alpha}{2}} + \sqrt{2 \cos^2 \alpha} + \sqrt{2 \cos^2 \frac{3\alpha}{2}} + \dots n \text{ terms} \\
 &= \sqrt{2} \left\{ \cos \frac{\alpha}{2} + \cos \alpha + \cos \frac{3\alpha}{2} + \dots + \text{to } n \text{ terms} \right\} \\
 &= \sqrt{2} \frac{\sin \frac{n\alpha}{4}}{\sin \frac{\alpha}{4}} \cdot \cos \frac{\frac{\alpha}{2} + \frac{\alpha}{2} + (n-1) \frac{\alpha}{2}}{2}
 \end{aligned}$$

$$= \sqrt{2} \frac{\sin \frac{n\alpha}{4}}{\sin \frac{\alpha}{4}} \cos \left[(n+1) \frac{\alpha}{4} \right]$$

$$\begin{aligned}
 5. \Sigma \cos^4 1^\circ - \Sigma \sin^4 1^\circ \\
 &= \Sigma (\cos^4 1^\circ - \sin^4 1^\circ) \\
 &= \Sigma \cos 2^\circ \\
 &= \cos 2^\circ + \cos 4^\circ + \cos 6^\circ + \dots + \cos 358^\circ \\
 &= \frac{\cos \left(\frac{2^\circ + 358^\circ}{2} \right) \sin(179^\circ \times 1^\circ)}{\sin 1^\circ} \\
 &= \frac{\cos 180^\circ \sin(180^\circ - 1^\circ)}{\sin 1^\circ} \\
 &= -1
 \end{aligned}$$

Exercise 3.8

$$\begin{aligned}
 1. (a) \text{L.H.S.} &= \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} \\
 &= \cos^2 \frac{A}{2} + \sin^2 \frac{C}{2} - \sin^2 \frac{B}{2} \\
 &= \cos^2 \frac{A}{2} + \sin \left(\frac{C-B}{2} \right) \sin \left(\frac{C+B}{2} \right) \\
 &= \cos^2 \frac{A}{2} + \sin \left(\frac{C-B}{2} \right) \cos \left(\frac{A}{2} \right) \\
 &= \cos \left(\frac{A}{2} \right) \left[\sin \left(\frac{B+C}{2} \right) + \sin \left(\frac{C-B}{2} \right) \right] \\
 &= 2 \cos \left(\frac{A}{2} \right) \cos \left(\frac{B}{2} \right) \sin \left(\frac{C}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 (b) \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \\
 &= \frac{1 + \cos A}{2} + \frac{1 + \cos B}{2} + \frac{1 + \cos C}{2} \\
 &= \frac{3 + \cos A + \cos B + \cos C}{2} \\
 &= \frac{3 + 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2} \\
 &= 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}
 \end{aligned}$$

$$\begin{aligned}
 2. (a) A + B + C &= \pi/2 \text{ or } 2A + 2B + 2C = \pi \\
 \text{Let } P &= 2A, Q = 2B, R = 2C \Rightarrow P + Q + R = \pi \\
 \sin^2 A + \sin^2 B + \sin^2 C \\
 &= \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2B}{2} + \frac{1 - \cos 2C}{2} \\
 &= \frac{3 - (\cos 2A + \cos 2B + \cos 2C)}{2} \\
 &= \frac{3 - (\cos P + \cos Q + \cos R)}{2} \\
 &= \frac{3 - \left(1 + 4 \sin \frac{P}{2} \sin \frac{Q}{2} \sin \frac{R}{2} \right)}{2} \\
 &= 1 - 2 \sin A \sin B \sin C
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \cos^2 A + \cos^2 B + \cos^2 C &= \frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} \\
 &= \frac{3 + (\cos 2A + \cos 2B + \cos 2C)}{2} \\
 &= \frac{3 + (\cos P + \cos Q + \cos R)}{2} \\
 &= \frac{3 + \left(1 + 4 \sin \frac{P}{2} \sin \frac{Q}{2} \sin \frac{R}{2}\right)}{2} \\
 &= 2 + 2 \sin A \sin B \sin C
 \end{aligned}$$

$$\begin{aligned}
 3. \text{ (a) L.H.S.} &= \cos^2 A + \cos^2 B + \cos^2 C \\
 &= \frac{3 + (\cos 2A + \cos 2B + \cos 2C)}{2} \\
 \text{Now given } A + B &= C \\
 \text{or } A + B + (\pi - C) &= \pi \\
 \text{or } A + B + D = \pi, \text{ where } D = \pi - C \\
 \Rightarrow \text{L.H.S.} &= \frac{3 + (\cos 2A + \cos 2B + \cos 2D)}{2} \\
 &= \frac{3 + (\cos 2A + \cos 2B + \cos 2D)}{2} \\
 &= \frac{3 + (-1 - 4 \cos A \cos B \cos D)}{2} \\
 &= 1 - 2 \cos A \cos B \cos D \\
 &= 1 - 2 \cos A \cos B \cos (\pi - C) \\
 &= 1 + 2 \cos A \cos B \cos C = \text{R.H.S.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) Given } \alpha + \beta &= 60^\circ \\
 \cos^2 \alpha + \cos^2 \beta - \cos \alpha \cos \beta &= 1 - \sin^2 \alpha + \cos^2 \beta - \cos \alpha \cos \beta \\
 &= 1 + \cos(\alpha + \beta) \cos(\alpha - \beta) - \cos \alpha \cos \beta \\
 &= 1 + \cos 60^\circ \cos(\alpha - \beta) - \cos \alpha \cos \beta \\
 &= 1 + \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{2} - \cos \alpha \cos \beta \\
 &= 1 - \frac{\cos(\alpha + \beta)}{2} \\
 &= 1 - \frac{1}{4} = \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ Let } A = \beta - \gamma, B = \gamma - \alpha, \text{ and } C = \pi - \alpha - \beta \\
 \Rightarrow A + B + C = \pi \\
 \Rightarrow \cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta) \\
 = \cos^2 A + \cos^2 B + \cos^2 C \\
 = \frac{3 + (\cos 2A + \cos 2B + \cos 2C)}{2} \\
 = \frac{3 + (-1 - 4 \cos A \cos B \cos C)}{2} \\
 = 1 - 2 \cos A \cos B \cos C \\
 = 1 + 2 \cos(\beta - \alpha) \cos(\gamma - \alpha) \cos(\alpha - \beta)
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ (a) } A + B + C &= (\pi/2) \Rightarrow A + B = (\pi/2) - C \\
 \Rightarrow \tan(A + B) &= \tan(\pi/2 - C) \\
 \Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} &= \cot C \quad \dots(i)
 \end{aligned}$$

Now convert all the terms in cot and simplify.

$$\text{(b) Replace } \cot C \text{ by } \frac{1}{\tan C} \text{ in Eq. (i) and simplify.}$$

$$\begin{aligned}
 6. \text{ (a) } A + B + C &= \pi \text{ or } 3A + 3B + 3C = 3\pi \\
 \Rightarrow P + Q + R &= 3\pi, \\
 \text{where } P = 3A, Q = 3B, R = 3C \\
 \Rightarrow \tan P + \tan Q + \tan R &= \tan P \tan Q \tan R \\
 \Rightarrow \tan 3A + \tan 3B + \tan 3C &= \tan 3A \tan 3B \tan 3C
 \end{aligned}$$

$$\text{(b) In triangle } ABC, \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$

$$\text{or } \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\text{or } \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\text{or } \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = \cot \frac{C}{2}$$

$$\text{or } \frac{\cot \frac{B}{2} + \cot \frac{A}{2}}{\cot \frac{A}{2} \cot \frac{B}{2} - 1} = \cot \frac{C}{2}$$

$$\text{or } \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

$$\begin{aligned}
 7. \frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} \\
 = \frac{-\cos(B + C)}{\sin B \sin C} + \frac{-\cos(A + C)}{\sin C \sin A} + \frac{-\cos(A + B)}{\sin A \sin B} \\
 = -[\cot B \cot C + \cot A \cot C + \cot A \cot B - 3] \\
 = -[1 - 3] = 2
 \end{aligned}$$

$$\begin{aligned}
 8. \cos 3A + \cos 3B + \cos 3C &= 1 \\
 \Rightarrow 2 \cos \frac{3(A + B)}{2} \cos \frac{3(A - B)}{2} &= 2 \sin^2 \frac{3C}{2} \\
 \Rightarrow -\sin \frac{3C}{2} \cos \frac{3(A - B)}{2} &= \sin^2 \frac{3C}{2} \\
 \Rightarrow \sin \frac{3C}{2} \left[\cos \frac{3(A - B)}{2} - \cos \frac{3(A + B)}{2} \right] &= 0 \\
 \Rightarrow 2 \sin \frac{3A}{2} \sin \frac{3B}{2} \sin \frac{3C}{2} &= 0
 \end{aligned}$$

i.e., one of the angles must be equal to 120° .

$$\begin{aligned}
 9. \text{ We have } 2 \sin^2 C &= 2 + \cos 2A + \cos 2B \\
 \therefore \sin^2 A + \sin^2 B + \sin^2 C &= 2 \\
 \text{Now in } \triangle ABC, \\
 \sin^2 A + \sin^2 B + \sin^2 C &= 2 + 2 \cos A \cos B \cos C \\
 \therefore \cos A \cos B \cos C &= 0 \\
 \text{Therefore, } \triangle ABC &\text{ must be right angled triangle.}
 \end{aligned}$$

Exercise 3.9

$$\begin{aligned}
 1. \text{ Let } x &= r \cos \theta \text{ and } y = r \sin \theta \\
 \Rightarrow E &= \frac{1}{4 \sin^2 \theta + \sin \theta \cos \theta + \cos^2 \theta} \\
 &= \frac{2}{5 + \sin 2\theta - 3 \cos 2\theta}
 \end{aligned}$$

$$\text{Now, } 5 - \sqrt{10} \leq 5 + \sin 2\theta - 3 \cos 2\theta \leq 5 + \sqrt{10}$$

$$\begin{aligned}
 \text{Hence, the maximum value of expression is } \frac{2}{5 - \sqrt{10}} \text{ and the} \\
 \text{minimum value is } \frac{2}{5 + \sqrt{10}}.
 \end{aligned}$$

Let $a = \sqrt{2} \cos \theta$, $b = \sqrt{2} \sin \theta$
 $\alpha = \sqrt{2} \cos \phi$, $\beta = \sqrt{2} \sin \phi$
 $S = 2 - 2 [\sin (\theta + \pi/4) + \sin (\phi + \pi/4)] + 2 [\cos (\theta - \phi)]$
 $\Rightarrow$ Maximum value occurs when $\theta = \phi = 5\pi/4$.
 $\Rightarrow S_{\max} = 2 - 2[-1 - 1] + 2 = 8$

Let $x = r \cos \theta$ and $y = r \sin \theta$.
 $x^2 + 2y^2 + 2xy = 1$
 $r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + 2r^2 \sin \theta \cos \theta = 1$

$$r^2 = \frac{1}{\cos^2 \theta + 2 \sin^2 \theta + \sin 2\theta}$$

$$= \frac{1}{\sin^2 \theta + 1 + \sin 2\theta}$$

$$= \frac{2}{1 - \cos 2\theta + 2 + 2 \sin 2\theta}$$

$$= \frac{2}{3 - \cos 2\theta + 2 \sin 2\theta}$$

Now, $-\sqrt{5} \leq -\cos 2\theta + 2 \sin 2\theta \leq \sqrt{5}$
 $\Rightarrow 3 - \sqrt{5} \leq 3 - \cos 2\theta + 2 \sin 2\theta \leq 3 + \sqrt{5}$
 $\Rightarrow r_{\max}^2 = \frac{2}{3 - \sqrt{5}}$

4. We have $\left(\frac{x}{12}\right)^2 - \left(\frac{y}{5}\right)^2 = 1$

Put $x = 12 \sec \theta$ and $y = 5 \tan \theta$

$$\therefore \frac{144}{x} + \frac{25}{y} = 12 \cos \theta + 5 \sin \theta$$

$$12 \cos \theta + 5 \sin \theta \in \left[-\sqrt{12^2 + 5^2}, \sqrt{12^2 + 5^2}\right] \text{ or } [-13, 13]$$

$$\therefore \frac{144}{x} + \frac{25}{y} \in [-13, 13]$$

5. We have $(x+3)^2 + (y-2)^2 = 5^2$

Let $x = -3 + 5 \cos \theta$ and $y = 2 + 5 \sin \theta$

$$\therefore 2x + y = -6 + 2 + 10 \cos \theta + 5 \sin \theta$$

$$= -4 + 5(2 \cos \theta + \sin \theta)$$

Now, $(2 \cos \theta + \sin \theta) \in [-\sqrt{5}, \sqrt{5}]$

$$\therefore 2x + y = -4 + 5(2 \cos \theta + \sin \theta) \in [-5\sqrt{5}, -4, 5\sqrt{5} - 4]$$

Exercise 3.10

1. (a) In $\triangle ABC$, we know that

$$\cos A + \cos B + \cos C \leq \frac{3}{2} \quad \dots(1)$$

$$\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}$$

$$= \frac{1 + \cos A}{2} + \frac{1 + \cos B}{2} + \frac{1 + \cos C}{2}$$

$$= \frac{3}{2} + \frac{\cos A + \cos B + \cos C}{2} \leq \frac{3}{2} + \frac{3}{4} \quad (\text{using Eq. (1)})$$

$$\therefore \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \leq \frac{9}{4}$$

(b) $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = y \left(x^2 + \frac{1}{x^2} \right)$

$$\therefore y \left(x^2 + \frac{1}{x^2} \right) \leq \frac{9}{4}$$

$$\therefore y \leq \frac{9}{4 \left(x^2 + \frac{1}{x^2} \right)}$$

Now $x^2 + \frac{1}{x^2} \geq 2$

$$\therefore y \leq \frac{9}{8}$$

Thus, maximum value of y is $\frac{9}{8}$.

2. Let $p = \tan \alpha \tan \beta$, $q = \tan \beta \tan \gamma$ and $r = \tan \alpha \tan \gamma$

Now, $(\sqrt{p} + \sqrt{q} + \sqrt{r})^2 = p + q + r + 2\sqrt{pq} + 2\sqrt{qr} + 2\sqrt{rp}$

Also, using G.M. $\leq$ A.M., we have

$$2\sqrt{pq} \leq p + q, 2\sqrt{qr} \leq q + r, 2\sqrt{rp} \leq r + p$$

$$\therefore (\sqrt{p} + \sqrt{q} + \sqrt{r})^2 \leq p + q + r + p + q + r + r + p$$

$$\therefore \sqrt{p} + \sqrt{q} + \sqrt{r} \leq \sqrt{3} \cdot \sqrt{p + q + r}$$

Now for $\alpha + \beta + \gamma = \frac{\pi}{2}$, we have

$$p + q + r = \tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \alpha \tan \gamma = 1$$

$$\therefore \sqrt{p} + \sqrt{q} + \sqrt{r} < \sqrt{3}$$

3. We know that in acute angled triangle ABC ,

$$\tan A \tan B \tan C \geq 3\sqrt{3} \quad \dots(1)$$

Now using A.M. $\geq$ G.M., we have

$$\frac{\tan^2 A + \tan^2 B + \tan^2 C}{3} \geq [\tan^2 A \cdot \tan^2 B \cdot \tan^2 C]^{1/3}$$

$$\therefore \tan^2 A + \tan^2 B + \tan^2 C \geq 3(27)^{1/3} \quad (\text{using (1)})$$

$$\Rightarrow \tan^2 A + \tan^2 B + \tan^2 C \geq 9$$

4. (a) We have already proved that in $\triangle ABC$,

$$\sin^2 A + \sin^2 B + \sin^2 C \leq \frac{9}{4} \quad \dots(1)$$

Now, using, A.M. $\geq$ G.M., we have

$$\therefore \frac{\sin^2 A + \sin^2 B + \sin^2 C}{3} \geq \sqrt[3]{\sin^2 A \sin^2 B \sin^2 C}$$

Therefore, from Eq. (1),

$$\left(\frac{9}{4}\right) \geq \frac{\sin^2 A + \sin^2 B + \sin^2 C}{3}$$

$$\therefore \frac{3}{4} \geq (\sin A \sin B \sin C)^{\frac{2}{3}}$$

$$\text{or } \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$$

(b) We have already proved that in $\triangle ABC$,

$$\sin A \sin B \sin C \leq 3 \frac{\sqrt{3}}{8} \quad \dots(2)$$

Now, $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

$$\leq 4 \left(3 \frac{\sqrt{3}}{8} \right) \quad (\text{using Eq. (2)})$$

$$\therefore \sin 2A + \sin 2B + \sin 2C \leq \frac{3\sqrt{3}}{2}$$

5. We know that in triangle ABC ,

$$\cos A + \cos B + \cos C \leq 3/2$$

Now, $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$$\therefore 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{3}{2} \quad \dots(1)$$

$$\text{So, } \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{1}{8}$$

Using A.M $\geq$ G.M., we get

$$\frac{\operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2}}{3} \geq \left(\operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \right)^{1/3}$$

$$\Rightarrow \operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2} \geq 3 \left(\frac{1}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \right)^{1/3}$$

$$\Rightarrow \operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2} \geq 3(8)^{1/3} \quad [\text{Using (1)}]$$

$$\Rightarrow \operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2} \geq 6$$

Exercises

Single Correct Answer Type

$$1. (1) \cos(A - B) = \frac{3}{5} \quad \dots(i)$$

$$\text{or } 5 \cos A \cos B + 5 \sin A \sin B = 3$$

From the second relation, we have

$$\sin A \sin B = 2 \cos A \cos B \quad \dots(ii)$$

$$\Rightarrow \cos A \cos B = \frac{1}{5} \text{ and } \sin A \sin B = \frac{2}{5}$$

$$2. (1) A = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \sin 90^\circ$$

$$B = \sqrt{2} \left[\frac{1}{\sqrt{2}} \sin 44^\circ + \frac{1}{\sqrt{2}} \cos 44^\circ \right]$$

$$= \sqrt{2} \sin (45^\circ + 44^\circ)$$

$$= \sqrt{2} \sin 89^\circ < \sqrt{2} \sin 90^\circ$$

$$= \sqrt{2} \therefore A > B$$

$$3. (4) \tan(100^\circ + 125^\circ) = \frac{\tan 100^\circ + \tan 125^\circ}{1 - \tan 100^\circ \tan 125^\circ}$$

$$\therefore \tan 225^\circ = \frac{\tan 100^\circ + \tan 125^\circ}{1 - \tan 100^\circ \tan 125^\circ}$$

$$\text{i.e., } 1 = \frac{\tan 100^\circ + \tan 125^\circ}{1 - \tan 100^\circ \tan 125^\circ}$$

$$\text{i.e., } \tan 100^\circ + \tan 125^\circ + \tan 100^\circ \tan 125^\circ = 1$$

$$4. (4) \cot(\alpha + \beta) = 0 \Rightarrow \cos(\alpha + \beta) = 0$$

$$\Rightarrow \sin(\alpha + 2\beta) = \sin(\alpha + \beta) \cos \beta + \sin \beta \cos(\alpha + \beta) \\ = \pm \cos \beta \quad (\because \sin(\alpha + \beta) = \pm 1)$$

$$5. (2) 3 \sin A \cos B = \sin B \cos A$$

$$\Rightarrow \cos A \sin B = \frac{3}{4}$$

$$\Rightarrow \sin(A + B) = 1$$

$$\Rightarrow C = \frac{\pi}{2}, B = \frac{\pi}{2} - A$$

$$\Rightarrow 3 \tan A = \tan \left(\frac{\pi}{2} - A \right)$$

$$\Rightarrow 3 = \cot^2 A$$

$$6. (1) \frac{a}{\sin(\theta - \alpha)} = \frac{b}{\sin(\theta - \beta)} = k_1 \\ \frac{c}{\cos(\theta - \alpha)} = \frac{d}{\cos(\theta - \beta)} = k_2$$

$$\Rightarrow \frac{ac + bd}{ad + bc} = \frac{k_1 k_2}{k_1 k_2} \cdot \frac{\sin(\theta - \alpha) \cos(\theta - \beta)}{\sin(\theta - \alpha) \cos(\theta - \beta)} \\ = \frac{\sin 2(\theta - \alpha) + \sin 2(\theta - \beta)}{2 \sin(2\theta - (\alpha + \beta))} \\ = \frac{2 \sin(2\theta - (\alpha + \beta)) \cos(\alpha - \beta)}{2 \sin(2\theta - (\alpha + \beta))} = \cos(\alpha - \beta)$$

$$7. (1) 2 \sin \frac{A}{2} \operatorname{cosec} \frac{B}{2} \left(\sin \frac{C}{2} - \cos \frac{A}{2} \cos \frac{B}{2} \right) - \cos A \\ = 2 \sin \frac{A}{2} \operatorname{cosec} \frac{B}{2} \times \left(\cos \frac{A+B}{2} - \cos \frac{A}{2} \cos \frac{B}{2} \right) - \cos A \\ = 2 \sin \frac{A}{2} \operatorname{cosec} \frac{B}{2} \left(-\sin \frac{A}{2} \sin \frac{B}{2} \right) - \cos A \\ = -2 \sin^2 \frac{A}{2} - \cos A = -1$$

$$8. (1) f(x) = 3 \cos x + 5 \sin(x - \pi/6)$$

$$= \frac{1}{2} \cos x + 5 \times \frac{\sqrt{3}}{2} \sin x$$

$$\text{Then, } -\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \leq f(x) \leq \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2}$$

$$\text{or } -\sqrt{19} \leq f(x) \leq \sqrt{19}$$

9. (2) We have

$$\frac{x}{\cos \theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$$

Therefore, each ratio is equal to

$$\frac{x + y + z}{\cos \theta + \cos\left(\theta - \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{2\pi}{3}\right)} = \frac{x + y + z}{0}$$

$$\text{or } x + y + z = 0$$

$$10. (2) \frac{1}{\sin 1^\circ} \left[\frac{\sin(1^\circ - 0^\circ)}{\cos 0^\circ \cos 1^\circ} + \frac{\sin(2^\circ - 1^\circ)}{\cos 1^\circ \cos 2^\circ} + \frac{\sin(3^\circ - 2^\circ)}{\cos 2^\circ \cos 3^\circ} + \dots \right. \\ \left. + \frac{\sin(45^\circ - 44^\circ)}{\cos 44^\circ \cos 45^\circ} \right]$$

$$= \frac{1}{\sin 1^\circ} [\tan 1^\circ + (\tan 2^\circ - \tan 1^\circ) + (\tan 3^\circ - \tan 2^\circ) \\ + (\tan 4^\circ - \tan 3^\circ) + \dots + (\tan 45^\circ - \tan 44^\circ)]$$

$$= \frac{1}{\sin 1^\circ} = \frac{1}{x}$$

$$11. (4) (\alpha - \beta) = (\theta - \beta) - (\theta - \alpha)$$

$$\Rightarrow \cos(\alpha - \beta) = \cos(\theta - \beta) \cos(\theta - \alpha) + \sin(\theta - \beta) \sin(\theta - \alpha)$$

$$= \frac{y}{b} \times \frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}}$$

$$\Rightarrow \left[\frac{xy}{ab} - \cos(\alpha - \beta) \right]^2 = \left(1 - \frac{x^2}{a^2} \right) \left(1 - \frac{y^2}{b^2} \right)$$

$$\text{or } \frac{x^2 y^2}{a^2 b^2} + \cos^2(\alpha - \beta) - \frac{2xy}{ab} \cos(\alpha - \beta)$$

$$= 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} + \frac{x^2 y^2}{a^2 b^2}$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos(\alpha - \beta) = \sin^2(\alpha - \beta)$$

$$12. (4) \min(2 + \sin x - \cos x) = \min \left[2 + \sqrt{2} \sin \left(x - \frac{\pi}{4} \right) \right] = 2 - \sqrt{2}$$

$$13. (1) \text{ Let } \alpha = \frac{\pi - A}{4}, \beta = \frac{\pi - B}{4}, \gamma = \frac{\pi - C}{4}$$

$$\Rightarrow \alpha + \beta + \gamma = \frac{\pi}{2}$$

$$\Rightarrow \sum \tan \alpha \tan \beta = 1$$

$$\Rightarrow \sum \tan^2 \alpha = 1 = \sum \tan \alpha \tan \beta$$

$$\Rightarrow \tan \alpha = \tan \beta = \tan \gamma$$

$$14. (1) (1 + \tan A)(1 + \tan B) = 2$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan(A + B) = 1, \text{ i.e., } A + B = \frac{\pi}{4}$$

$$\text{or } \alpha + 4\alpha = \frac{\pi}{4}, \text{ i.e., } \alpha = \frac{\pi}{20}$$

$$15. (1) \cos 17^\circ = \cos(45^\circ - 28^\circ) \\ = \cos 45^\circ \cos 28^\circ + \sin 45^\circ \sin 28^\circ \\ = \frac{\cos 28^\circ + \sin 28^\circ}{\sqrt{2}} = \frac{k^3}{\sqrt{2}}$$

$$16. (1) f(\beta) = f\left(\frac{5\pi}{4} - \alpha\right) = \frac{\cot\left(\frac{5\pi}{4} - \alpha\right)}{1 + \cot\left(\frac{5\pi}{4} - \alpha\right)} \\ = \frac{1}{1 + \tan\left(\frac{5\pi}{4} - \alpha\right)} \\ = \frac{1}{1 + \frac{1 - \tan \alpha}{1 + \tan \alpha}} = \frac{1 + \tan \alpha}{2}$$

$$\text{As } f(\alpha) = \frac{\cot \alpha}{1 + \cot \alpha} = \frac{1}{1 + \tan \alpha}, \text{ we have}$$

$$f(\alpha)f(\beta) = \frac{1}{2}$$

$$17. (3) A - B = \frac{\pi}{4} \text{ or } \tan(A - B) = \tan \frac{\pi}{4}$$

$$\text{or } \frac{\tan A - \tan B}{1 + \tan A \tan B} = 1$$

$$\text{or } \tan A - \tan B - \tan A \tan B = 1$$

$$\text{or } \tan A - \tan B - \tan A \tan B + 1 = 2$$

$$\text{or } (1 + \tan A)(1 - \tan B) = 2 \Rightarrow y = 2$$

$$\text{Hence, } (y + 1)^{y+1} = (2 + 1)^{2+1} = (3)^3 = 27.$$

$$18. (4) \frac{\sin x}{\sin y} = \frac{1}{2}, \frac{\cos x}{\cos y} = \frac{3}{2}$$

$$\Rightarrow \frac{\tan x}{\tan y} = \frac{1}{3}$$

$$\Rightarrow \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{4 \tan x}{1 - 3 \tan^2 x}$$

$$\text{Also } \sin y = 2 \sin x, \cos y = \frac{2}{3} \cos x$$

$$\text{or } \sin^2 y + \cos^2 y = 4 \sin^2 x + \frac{4 \cos^2 x}{9} = 1$$

$$\text{or } 36 \tan^2 x + 4 = 9 \sec^2 x = 9(1 + \tan^2 x)$$

$$\text{or } 27 \tan^2 x = 5$$

$$\text{or } \tan x = \frac{\sqrt{5}}{3\sqrt{3}}$$

$$\Rightarrow \tan(x + y) = \frac{\frac{4\sqrt{5}}{3\sqrt{3}}}{1 - \frac{15}{27}} = \frac{4\sqrt{5} \times 27}{12 \times 3\sqrt{3}} = \sqrt{15}$$

$$19. (2) \cot^2 x = \cot(x - y) \cot(x - z)$$

$$\text{or } \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x} \right) \left(\frac{\cot x \cot z + 1}{\cot z - \cot x} \right)$$

$$\text{or } \cot^2 x \cot y \cot z - \cot^3 x \cot y - \cot^3 x \cot z + \cot^4 x \\ = \cot^2 x \cot y \cot z + \cot x \cot y + \cot x \cot z + 1$$

$$\text{or } \cot^3 x (\cot y + \cot z) + \cot x (\cot y + \cot z) + 1 - \cot^4 x = 0$$

$$\text{or } \cot x (\cot y + \cot z) (1 + \cot^2 x) + (1 - \cot^2 x) (1 + \cot^2 x) = 0$$

$$\text{or } [\cot x (\cot y + \cot z) + (1 - \cot^2 x)] = 0$$

$$\text{or } \frac{\cot^2 x - 1}{2 \cot x} = \frac{1}{2} (\cot y + \cot z)$$

$$20. (2) \text{ Let } \tan A = 3k, \tan B = 4k \text{ and } \tan C = 5k$$

$$\text{Now, } \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C,$$

$$\Rightarrow 12k = 60k^3$$

$$\Rightarrow k = \frac{1}{\sqrt{5}}$$

$$\therefore \tan A = \frac{3}{\sqrt{5}}, \tan B = \frac{4}{\sqrt{5}} \text{ and } \tan C = \sqrt{5}$$

$$\therefore \sin A = \frac{3}{\sqrt{14}}, \sin B = \frac{4}{\sqrt{21}}, \sin C = \frac{\sqrt{5}}{\sqrt{6}}$$

$$\text{Hence, } \sin A \cdot \sin B \cdot \sin C = \frac{2\sqrt{5}}{7}$$

$$21. (2) \sin 27^\circ - \sin 63^\circ = -2 \cos 45^\circ \sin 18^\circ$$

$$= -\sqrt{2} \left(\frac{\sqrt{5} - 1}{4} \right) = -\frac{\sqrt{5} - 1}{2\sqrt{2}} = -\frac{\sqrt{3} - \sqrt{5}}{2}$$

$$22. (2) \tan \left(\frac{\theta_1 - \theta_2}{2} \right) \tan \left(\frac{\theta_1 + \theta_2}{2} \right)$$

$$= \frac{2 \sin \left(\frac{\theta_1 + \theta_2}{2} \right) \sin \left(\frac{\theta_1 - \theta_2}{2} \right)}{2 \cos \left(\frac{\theta_1 + \theta_2}{2} \right) \cos \left(\frac{\theta_1 - \theta_2}{2} \right)}$$

$$= \frac{\cos \theta_2 - \cos \theta_1}{\cos \theta_1 + \cos \theta_2} = \frac{-1}{3}$$

$$23. (1) \text{ We have } \sin \alpha + \sin \beta = -\frac{21}{65} \quad \dots(i)$$

$$\cos \alpha + \cos \beta = -\frac{27}{65} \quad \dots(ii)$$

Squaring Eq. (i), we get

$$\sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta = \left(\frac{21}{65}\right)^2 \quad \dots(iii)$$

Squaring Eq. (ii), we get

$$\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta = \left(\frac{27}{65}\right)^2 \quad \dots(iv)$$

Adding Eqs. (iii) and (iv), we get

$$\begin{aligned} 2 + 2 \cos(\alpha - \beta) &= \frac{1}{(65)^2} [(27)^2 + (21)^2] \\ &= \frac{1}{(65)^2} (729 + 441) \\ &= \frac{1}{(65)^2} (1170) = \frac{18}{65} \end{aligned}$$

$$\text{or } 1 + \cos(\alpha - \beta) = \frac{9}{65}$$

$$\text{or } 2 \cos^2 \frac{\alpha - \beta}{2} = \frac{9}{65}$$

$$\text{or } \cos \frac{\alpha - \beta}{2} = -\frac{3}{\sqrt{130}}$$

$$\left[\because \pi < \alpha - \beta < 3\pi \Rightarrow \frac{\pi}{2} < \frac{\alpha - \beta}{2} < \frac{3\pi}{2} \Rightarrow \cos\left(\frac{\alpha - \beta}{2}\right) < 0 \right]$$

$$24. (1) \alpha + 2\alpha + 4\alpha = 7\alpha = \frac{\pi}{2}$$

$$\therefore \tan \alpha \cdot \tan 2\alpha + \tan 2\alpha \cdot \tan 4\alpha + \tan \alpha \cdot \tan 4\alpha = 1$$

$$25. (3) \frac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta}$$

$$= \frac{(\sin 3\theta + \sin 9\theta) + (\sin 5\theta + \sin 7\theta)}{(\cos 3\theta + \cos 9\theta) + (\cos 5\theta + \cos 7\theta)}$$

$$= \frac{2 \sin 6\theta \cos 3\theta + 2 \sin 6\theta \cos \theta}{2 \cos 6\theta \cos 3\theta + 2 \cos 6\theta \cos \theta}$$

$$= \frac{2 \sin 6\theta (\cos 3\theta + \cos \theta)}{2 \cos 6\theta (\cos 3\theta + \cos \theta)}$$

$$= \tan 6\theta$$

$$26. (2) \frac{\sin x - \sin z}{\cos z - \cos x} = \frac{2 \cos\left(\frac{x+z}{2}\right) \sin\left(\frac{x-z}{2}\right)}{2 \sin\left(\frac{x+z}{2}\right) \sin\left(\frac{x-z}{2}\right)}$$

$$= \cot\left(\frac{x+z}{2}\right) = \cot(y)$$

27. (3) For each of the ratios be $1/k$,

$$\frac{a+c}{b+d} = \frac{k \cos x + k \cos(x+2\theta)}{k \cos(x+\theta) + k \cos(x+3\theta)}$$

$$= \frac{2 \cos(x+\theta) \cos \theta}{2 \cos(x+2\theta) \cos \theta}$$

$$= \frac{\cos(x+\theta)}{\cos(x+2\theta)} = \frac{k \cos(x+\theta)}{k \cos(x+2\theta)} = \frac{b}{c}$$

$$28. (2) (\cos \alpha + \cos \beta)^2 - (\sin \alpha + \sin \beta)^2 = 0$$

$$\text{or } (\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta) -$$

$$(\sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta) = 0$$

$$\text{or } \cos 2\alpha + \cos 2\beta = -2(\cos \alpha \cos \beta - \sin \alpha \sin \beta) = -2 \cos(\alpha + \beta)$$

$$29. (2) \frac{\left[3 + \frac{\cos 80^\circ \cos 20^\circ}{\sin 80^\circ \sin 20^\circ}\right]}{\left[\frac{\cos 80^\circ}{\sin 80^\circ} + \frac{\cos 20^\circ}{\sin 20^\circ}\right]}$$

$$= \frac{[2 \sin 80^\circ \sin 20^\circ + (\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ)]}{\sin 20^\circ \cos 80^\circ + \cos 20^\circ \sin 80^\circ}$$

$$= \frac{-\cos 100^\circ + \cos 60^\circ + \cos 60^\circ}{\sin 100^\circ}$$

$$= \frac{1 - \cos 100^\circ}{\sin 100^\circ}$$

$$= \tan 50^\circ$$

$$30. (4) \text{ We have } 4x^2 - 16x + 15 < 0$$

$$\Rightarrow \frac{3}{2} < x < \frac{5}{2}$$

Therefore, the integral solution of

$$4x^2 - 16x + 15 < 0 \text{ is } x = 2$$

Thus, $\tan \alpha = 2$. It is given that $\cos \beta = \tan 45^\circ = 1$.

$$\therefore \sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$$

$$= \frac{1}{1 + \cot^2 \alpha} - (1 - \cos^2 \beta)$$

$$= \frac{1}{1 + \frac{1}{4}} - 0 = \frac{4}{5}$$

$$31. (2) f(n) = 2 \cos nx$$

$$\begin{aligned} \Rightarrow f(1) f(n+1) - f(n) \\ &= 4 \cos x \cos(n+1)x - 2 \cos nx \\ &= 2[2 \cos(n+1)x \cos x - \cos nx] \\ &= 2[\cos(n+2)x + \cos nx - \cos nx] \\ &= 2 \cos(n+2)x = f(n+2) \end{aligned}$$

$$32. (2) \sin \theta_1 - \sin \theta_2 = a, \cos \theta_1 + \cos \theta_2 = b$$

$$\Rightarrow a^2 + b^2 = 2 + 2 \cos(\theta_1 + \theta_2)$$

$$\Rightarrow 0 \leq a^2 + b^2 \leq 4$$

$$33. (3) \frac{\sqrt{2} - \sin \alpha - \cos \alpha}{\sin \alpha - \cos \alpha}$$

$$= \frac{\sqrt{2} - \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{\sqrt{2}} \cos \alpha \right)}{\sqrt{2} \left(\frac{1}{\sqrt{2}} \sin \alpha - \frac{1}{\sqrt{2}} \cos \alpha \right)}$$

$$= \frac{\sqrt{2} - \sqrt{2} \cos\left(\alpha - \frac{\pi}{4}\right)}{\sqrt{2} \sin\left(\alpha - \frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2} (1 - \cos \theta)}{\sqrt{2} \sin \theta},$$

$$\text{where } \theta = \alpha - \frac{\pi}{4} = \frac{2 \sin^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)}$$

$$= \tan \frac{\theta}{2} = \tan \left(\frac{\alpha}{2} - \frac{\pi}{8} \right)$$

$$\Rightarrow \frac{a \cos x + b \sin x = c}{a \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + \frac{2b \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = c}$$

$$\Rightarrow (c+a) \tan^2 \frac{x}{2} - 2b \tan \frac{x}{2} + c - a = 0$$

$$\Rightarrow \tan \frac{x_1}{2} + \tan \frac{x_2}{2} = \frac{2b}{c+a}$$

$$\text{and } \tan \frac{x_1}{2} \tan \frac{x_2}{2} = \frac{c-a}{c+a}$$

$$\Rightarrow \tan \left(\frac{x_1 + x_2}{2} \right) = \frac{\frac{2b}{c+a}}{1 - \frac{c-a}{c+a}} = \frac{2b}{2a} = \frac{b}{a}$$

Applying $b-a = c-b$ for A.P., we get
 $2 \cos z \sin(x-y) = 2 \cos x \sin(y-z)$
 Dividing by $2 \cos x \cos y \cos z$, etc., we get
 $\tan x - \tan y = \tan y - \tan z$

$$\frac{\tan(\alpha + \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \frac{\tan \gamma}{\tan \beta}$$

$$\Rightarrow \frac{\sin(\alpha + \beta - \gamma) \cos(\alpha - \beta + \gamma)}{\sin(\alpha - \beta + \gamma) \cos(\alpha + \beta - \gamma)} = \frac{\sin \gamma \cos \beta}{\sin \beta \cos \gamma}$$

Applying componendo and dividendo, we get

$$\frac{\sin 2\alpha}{\sin 2(\beta - \gamma)} = \frac{\sin(\gamma + \beta)}{\sin(\gamma - \beta)}$$

$$\Rightarrow \sin 2(\beta - \gamma) \sin(\beta + \gamma) + \sin 2\alpha \sin(\beta - \gamma) = 0$$

$$\Rightarrow \sin(\beta - \gamma) (2 \cos(\beta - \gamma) \sin(\beta + \gamma) + \sin 2\alpha) = 0$$

$$\Rightarrow \sin(\beta - \gamma) (\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$$

$$\Rightarrow \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0 \quad (\text{as } \beta \neq \gamma)$$

$$\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 = -1$$

$$\text{or } \cos(\theta_1 + \theta_2) = 1$$

$$\Rightarrow \theta_1 + \theta_2 = 2n\pi, n \in I$$

$$\text{or } \frac{\theta_1}{2} + \frac{\theta_2}{2} = n\pi$$

$$\text{Thus, } \tan \frac{\theta_1}{2} \cot \frac{\theta_2}{2} = \tan \frac{\theta_1}{2} \cot \left(n\pi - \frac{\theta_1}{2} \right)$$

$$= -\tan \frac{\theta_1}{2} \cot \frac{\theta_1}{2} = -1$$

$$\text{Numerator} = 2[(\sin 1^\circ + \sin 89^\circ) + (\sin 2^\circ + \sin 88^\circ) + \dots + (\sin 44^\circ + \sin 46^\circ) + \sin 45^\circ]$$

$$\Rightarrow \frac{\text{Numerator}}{\text{Denominator}} = \frac{2[\sin 45^\circ \{2(\cos 44^\circ + \cos 43^\circ + \dots + \cos 1^\circ) + 1\}]}{2 \sin 45^\circ} = \sqrt{2}$$

$$\text{Given } \cot \frac{A}{2} \cdot \cot \frac{C}{2} = 3$$

$$\Rightarrow \frac{\cos \frac{A}{2} \cdot \cos \frac{C}{2}}{\sin \frac{A}{2} \cdot \sin \frac{C}{2}} = 3$$

$$\Rightarrow \frac{\cos \frac{A-C}{2}}{\cos \frac{A+C}{2}} = 2$$

(using componendo and dividendo)

$$\Rightarrow \frac{2 \sin \frac{A+C}{2} \cos \frac{A-C}{2}}{2 \sin \frac{A+C}{2} \cos \frac{A+C}{2}} = 2$$

$$\Rightarrow 2 \sin B = \sin A + \sin C$$

$$40. (2) 2 \sec 2\theta = \tan \phi + \cot \phi$$

$$\text{or } \frac{2}{\cos 2\theta} = \frac{\sin^2 \phi + \cos^2 \phi}{\sin \phi \cos \phi}$$

$$\text{or } \frac{2}{\cos 2\theta} = \frac{1}{\sin \phi \cos \phi}$$

$$\text{or } \cos 2\theta = \sin 2\phi$$

$$\text{or } 2\theta = 90^\circ - 2\phi$$

$$\text{or } \theta + \phi = \frac{\pi}{4}$$

$$41. (2) 4x^2 - 2\sqrt{5}x + 1 = 0$$

$$\therefore x = \frac{2\sqrt{5} \pm 2}{8} = \frac{\sqrt{5} \pm 1}{4}$$

$$\therefore \text{Roots are } \sin 18^\circ \text{ and } \cos 36^\circ.$$

$$42. (2) 3 \sin^2 A + 2 \sin^2 B = 1$$

$$\text{or } 3 \sin^2 A = \cos 2B$$

$$\text{Also } 3 \sin 2A - 2 \sin 2B = 0$$

$$\text{or } \sin 2B = \frac{3}{2} \sin 2A$$

$$\text{Now, } \cos(A + 2B) = \cos A \cos 2B - \sin A \sin 2B$$

$$= \cos A \cdot 3 \sin^2 A - \sin A \cdot \frac{3}{2} \sin 2A$$

$$= 3 \sin^2 A \cos A - 3 \sin^2 A \cos A = 0$$

$$\therefore A + 2B = \frac{\pi}{2}$$

$$43. (3) \text{ Squaring}$$

$$1 + 2 \sin 25^\circ \cdot \cos 25^\circ = p^2$$

$$\therefore \sin 50^\circ = p^2 - 1$$

$$\therefore \cos 50^\circ = \sqrt{1 - (p^2 - 1)^2} = p \sqrt{2 - p^2} \quad (\because p > 0)$$

$$44. (4) \tan \left(\frac{\pi}{2} - \frac{7\pi}{16} \right) + 2 \tan \left(\frac{\pi}{2} - \frac{3\pi}{8} \right) - \cot \left(\pi - \frac{15\pi}{16} \right)$$

$$= \tan \frac{\pi}{16} - \cot \frac{\pi}{16} + 2 \tan \frac{\pi}{8}$$

$$= \frac{\sin \frac{\pi}{16}}{\cos \frac{\pi}{16}} - \frac{\cos \frac{\pi}{16}}{\sin \frac{\pi}{16}} + 2 \tan \frac{\pi}{8}$$

$$= -\frac{2 \cos \frac{\pi}{8}}{\sin \frac{\pi}{8}} + 2 \frac{\sin \frac{\pi}{8}}{\cos \frac{\pi}{8}} = -\frac{4 \cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = -4$$

$$45. (2) \text{ Since } \alpha < \beta < \gamma < \delta \text{ and } \sin \alpha = \sin \beta = \sin \gamma = \sin \delta = k, \text{ we have}$$

$$\beta = \pi - \alpha, \gamma = 2\pi - \alpha, \delta = 3\pi - \alpha$$

$$\Rightarrow 4 \sin \frac{\alpha}{2} + 3 \sin \frac{\beta}{2} + 2 \sin \frac{\gamma}{2} + \sin \frac{\delta}{2}$$

$$= 4 \sin \frac{\alpha}{2} + 3 \cos \frac{\alpha}{2} - 2 \sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}$$

$$= 2 \sin \frac{\alpha}{2} + 2 \cos \frac{\alpha}{2} = 2\sqrt{1 + \sin \alpha} = 2\sqrt{1 + k}$$

$$\begin{aligned}
 46. (2) \quad \frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} &= \frac{2 \sin(A+B) \sin(A-B)}{\sin 2A - \sin 2B} \\
 &= \frac{2 \sin(A+B) \sin(A-B)}{2 \sin(A-B) \cos(A+B)} \\
 &= \tan(A+B)
 \end{aligned}$$

$$47. (4) \text{ Put, } \beta = 0 \Rightarrow \tan \alpha = \frac{1}{3}$$

$$\Rightarrow \sin 2\alpha = \frac{2\left(\frac{1}{3}\right)}{1 + \frac{1}{9}} = \frac{3}{5}$$

$$\begin{aligned}
 \therefore \frac{1}{1 - 3 \sin 2\alpha} + \frac{1}{1 - 3 \sin 2\beta} \\
 = \frac{1}{1 - 3 \sin 2\alpha} + 1 = \frac{1}{1 - \frac{9}{5}} + 1 = \frac{-1}{4}
 \end{aligned}$$

$$\begin{aligned}
 48. (3) \quad \cos^2 10^\circ - \cos 10^\circ \cos 50^\circ + \cos^2 50^\circ \\
 = \frac{1}{2} [1 + \cos 20^\circ - (\cos 60^\circ + \cos 40^\circ) \\
 + (1 + \cos 100^\circ)] \\
 = \frac{1}{2} [1 + \cos 20^\circ - \frac{1}{2} - \cos 40^\circ + 1 - \cos 80^\circ] \\
 = \frac{1}{2} \left[\frac{3}{2} + \cos 20^\circ - (2 \cos 60^\circ \cos 20^\circ) \right] = \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 49. (2) \quad \tan^2 \theta &= 2 \tan^2 \phi + 1 \\
 \text{or } 1 + \tan^2 \theta &= 2(1 + \tan^2 \phi) \\
 \Rightarrow \sec^2 \theta &= 2 \sec^2 \phi \\
 \Rightarrow \cos^2 \phi &= 2 \cos^2 \theta = 1 + \cos 2\theta \\
 \Rightarrow \cos 2\theta &= \cos^2 \phi - 1 = -\sin^2 \phi \\
 \Rightarrow \sin^2 \phi + \cos 2\theta &= 0
 \end{aligned}$$

$$50. (4) \tan A \cdot \tan B = \frac{1}{2}$$

$$\Rightarrow \frac{\tan A}{1} = \frac{\cot B}{2} = \lambda (\text{say})$$

$$\Rightarrow \tan A = \lambda \text{ and } \tan B = \frac{1}{2\lambda}$$

$$\therefore (5 - 3 \cos 2A)(5 - 3 \cos 2B)$$

$$= \left[5 - \frac{3(1 - \lambda^2)}{(1 + \lambda^2)} \right] \times \left[5 - \frac{3\left(1 - \frac{1}{4\lambda^2}\right)}{\left(1 + \frac{1}{4\lambda^2}\right)} \right] = 16$$

$$51. (2) \text{ It is true when } x = \frac{\pi}{2}, y = \frac{\pi}{4}.$$

$$\tan y = 1 \Rightarrow \frac{2 \tan \frac{y}{2}}{1 - \tan^2 \frac{y}{2}} = 1$$

$$\Rightarrow \tan^2 \frac{y}{2} + 2 \tan \frac{y}{2} = 1$$

$$52. (3) \quad 2|\sin 2\alpha| \leq 2$$

$$\text{and } |\tan \beta + \cot \beta| = \frac{1}{|\sin \beta \cos \beta|} = 2 |\operatorname{cosec} 2\beta| \geq 2$$

$$\therefore 2|\sin 2\alpha| = |\tan \beta + \cot \beta| = 2$$

$$\therefore \alpha = \beta = \frac{3\pi}{4}$$

$$53. (4) \text{ We have } \sin^3 10^\circ + \sin^3 50^\circ - \sin^3 70^\circ$$

$$\begin{aligned}
 &= \frac{1}{4} [(3 \sin 10^\circ - \sin 30^\circ) + (3 \sin 50^\circ - \sin 150^\circ) - (3 \sin 70^\circ - \sin 210^\circ)] \\
 &= \frac{1}{4} \left[3(\sin 10^\circ + \sin 50^\circ - \sin 70^\circ) - \frac{3}{2} \right] \\
 &= \frac{1}{4} \left[3(\sin 10^\circ - 2 \cos 60^\circ \cdot \sin 10^\circ) - \frac{3}{2} \right] = -\frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 54. (2) \quad P(x) &= \left(\frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} \right)^2 + \left(\frac{1 + \cot x + \cot^2 x}{1 + \tan x + \tan^2 x} \right) \\
 &= \left(\frac{2 \sin^2 x + 2 \sin x \cos x}{2 \cos^2 x + 2 \sin x \cos x} \right)^2 + \left(\frac{1}{\tan^2 x} \cdot \frac{\tan^2 x + \tan x + 1}{1 + \tan x + \tan^2 x} \right) \\
 &= \tan^2 x + \cot^2 x \\
 &= (\tan x - \cot x)^2 + 2
 \end{aligned}$$

$$55. (4) \text{ Let } \theta = \frac{\pi}{7}$$

$$\Rightarrow 3\theta = \pi - 4\theta$$

$$\Rightarrow \sin 3\theta = \sin 4\theta$$

$$\Rightarrow 3 \sin \theta - 4 \sin^3 \theta = 4 \sin \theta \cos \theta \cos 2\theta$$

$$\Rightarrow 3 - 4 \sin^2 \theta = 4 \cos \theta (2 \cos^2 \theta - 1)$$

$$\Rightarrow 8 \cos^3 \theta - 4 \cos \theta = 4 \cos^2 \theta - 1$$

$$\Rightarrow 4 \cos \theta (2 \cos^2 \theta - 1) = 4 \cos^2 \theta - 1$$

$$\Rightarrow 4 \cos \theta = \frac{4 \cos^2 \theta - 1}{2 \cos^2 \theta - 1} = \frac{3 - \tan^2 \theta}{1 - \tan^2 \theta}$$

$$56. (1) \text{ Let } \alpha = \frac{180^\circ}{7}$$

$$\Rightarrow 180^\circ = 7\alpha$$

$$\Rightarrow 3\alpha = 180^\circ - 4\alpha$$

$$\Rightarrow \sin 3\alpha = \sin 4\alpha$$

$$\therefore \operatorname{cosec} \frac{360^\circ}{7} + \operatorname{cosec} \frac{540^\circ}{7}$$

$$= \operatorname{cosec} 2\alpha + \operatorname{cosec} 3\alpha$$

$$= \frac{\sin 3\alpha + \sin 2\alpha}{\sin 3\alpha \sin 2\alpha}$$

$$= \frac{\sin 4\alpha + \sin 2\alpha}{\sin 3\alpha \cdot \sin 2\alpha}$$

$$= \frac{2 \sin 3\alpha \cos \alpha}{\sin 3\alpha \cdot \sin 2\alpha}$$

$$= \frac{2 \cos \alpha}{2 \sin \alpha \cdot \cos \alpha}$$

$$= \operatorname{cosec} \alpha$$

$$= \operatorname{cosec} \frac{180^\circ}{7}$$

from (i)

$$\text{or } \tan \theta = \lambda, \text{ we get } \frac{2 \tan \theta / 2}{1 - \tan^2 \theta / 2} = \lambda$$

$$\text{or } \lambda \tan^2 \frac{\theta}{2} + 2 \tan \frac{\theta}{2} - \lambda = 0$$

$$\Rightarrow \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = -1$$

$$\text{or } \tan \theta = \sqrt{n}$$

$$\Rightarrow \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - n}{1 + n} = \text{rational}$$

$$\text{or } \sin 2\theta = \cos 3\theta$$

$$\text{or } 2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\text{or } 2 \sin \theta = 4(1 - \sin^2 \theta) - 3$$

$$\text{or } 4 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\text{or } \sin \theta = \frac{\sqrt{5} - 1}{4}$$

$$\text{We have } \cos x = \tan y$$

$$\text{or } \cos^2 x = \tan^2 y$$

$$= \sec^2 y - 1$$

$$= \cot^2 z - 1 \quad [\because \cos y = \tan z, \sec y = \cot z]$$

$$\Rightarrow 1 + \cos^2 x = \cot^2 z = \frac{\tan^2 x}{1 - \tan^2 x} \quad [\because \cos z = \tan x]$$

$$= \frac{\sin^2 x}{\cos^2 x - \sin^2 x}$$

$$\Rightarrow 2 \sin^4 x - 6 \sin^2 x + 2 = 0$$

$$\text{or } \sin^2 x = \frac{3 - \sqrt{5}}{2} = \left(\frac{\sqrt{5} - 1}{2} \right)^2$$

$$\text{or } \sin x = \frac{\sqrt{5} - 1}{2} = 2 \sin 18^\circ$$

$$\text{61. (2) } \cot 70^\circ + 4 \cos 70^\circ = \frac{\cos 70^\circ + 4 \sin 70^\circ \cos 70^\circ}{\sin 70^\circ}$$

$$= \frac{\cos 70^\circ + 2 \sin 140^\circ}{\sin 70^\circ}$$

$$= \frac{\cos 70^\circ + 2 \sin(180^\circ - 40^\circ)}{\sin 70^\circ}$$

$$= \frac{\sin 20^\circ + \sin 40^\circ + \sin 40^\circ}{\sin 70^\circ}$$

$$= \frac{2 \sin 30^\circ \cos 10^\circ + \sin 40^\circ}{\sin 70^\circ}$$

$$= \frac{\sin 80^\circ + \sin 40^\circ}{\sin 70^\circ}$$

$$= \frac{2 \sin 60^\circ \cos 20^\circ}{\sin 70^\circ}$$

$$= \sqrt{3}$$

$$\text{62. (2) } \sin x + \cos x = \frac{\sqrt{7}}{2}$$

$$\Rightarrow \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{\sqrt{7}}{2}$$

$$\text{or } (\sqrt{7} + 2) \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2} + (\sqrt{7} - 2) = 0$$

$$\text{or } \tan \frac{x}{2} = \frac{4 \pm \sqrt{16 - 4(7 - 4)}}{2(\sqrt{7} + 2)} = \frac{1}{(\sqrt{7} + 2)} \quad \left(\text{as } \frac{x}{2} < \frac{\pi}{8} \right)$$

$$= \frac{\sqrt{7} - 2}{3}$$

$$\text{63. (4) } \frac{\tan 3A}{\tan A} = k (k \neq 1)$$

$$\therefore \frac{\sin 3A \cos A}{\sin A \cos 3A} = \frac{k}{1}$$

$$\therefore \frac{k + 1}{k - 1} = \frac{\sin 4A}{\sin 2A} = 2 \cos 2A \quad (\text{using comp. and divd.})$$

$$\therefore \frac{\sin 3A}{\sin A} = 1 + 2 \cos 2A = \frac{2k}{k - 1}$$

$$\text{Similarly, } \frac{\cos A}{\cos 3A} = \frac{k - 1}{2}$$

$$\text{64. (2) } 4 \cos^2 \left(\frac{\pi}{4} - \frac{x}{2} \right) + \sqrt{4 \sin^4 x + \sin^2 2x}$$

$$= 4 \cos^2 \left(\frac{\pi}{4} - \frac{x}{2} \right) + \sqrt{4 \sin^2 x (\cos^2 x + \sin^2 x)}$$

$$= 2 \left(1 + \cos \left(\frac{\pi}{2} - x \right) \right) + 2 |\sin x|$$

$$= 2 + 2 \sin x - 2 \sin x$$

$$= 2$$

$$\left[\text{as } x \in \left(\pi, \frac{3\pi}{2} \right) \right]$$

$$\text{65. (2) } \cos x = \frac{2 \cos y - 1}{2 - \cos y}$$

$$\Rightarrow \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2(1 - \tan^2 \frac{y}{2}) - 1}{2 - \frac{1 - \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2}}} = \frac{1 - 3 \tan^2 \frac{y}{2}}{1 + 3 \tan^2 \frac{y}{2}}$$

$$\Rightarrow \tan^2 \frac{x}{2} = 3 \tan^2 \frac{y}{2}$$

$$\text{or } \tan \frac{x}{2} \cot \frac{y}{2} = \sqrt{3} \quad (\because x, y \in (0, \pi))$$

$$\text{66. (3) } \frac{\cos 16^\circ \cos 44^\circ}{\sin 16^\circ \sin 44^\circ} - 1 + \frac{\cos 44^\circ \cos 76^\circ}{\sin 44^\circ \sin 76^\circ} - 1 - \frac{\cos 76^\circ \cos 16^\circ}{\sin 76^\circ \sin 16^\circ} - 1 + 3$$

$$= \frac{\cos 60^\circ}{\sin 16^\circ \sin 44^\circ} + \frac{\cos 120^\circ}{\sin 44^\circ \sin 76^\circ} - \frac{\cos 60^\circ}{\sin 76^\circ \sin 16^\circ} + 3$$

$$= \frac{1}{2} \left(\frac{\sin 76^\circ - \sin 16^\circ}{\sin 16^\circ \sin 44^\circ \sin 76^\circ} \right) - \frac{1}{2 \sin 76^\circ \sin 16^\circ} + 3$$

$$= \frac{1}{2} \left(\frac{2 \sin 30^\circ \cos 46^\circ}{\sin 16^\circ \sin 44^\circ \sin 76^\circ} \right) - \frac{1}{2 \sin 76^\circ \sin 16^\circ} + 3$$

$$= \frac{1}{2} \left(\frac{2 \sin 30^\circ \sin 44^\circ}{\sin 16^\circ \sin 44^\circ \sin 76^\circ} \right) - \frac{1}{2 \sin 76^\circ \sin 16^\circ} + 3$$

$$= 3$$

$$\text{67. (2) } \text{We have } \sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \frac{a+b+a-b}{\sqrt{(a^2-b^2)}}$$

$$= \frac{2a}{\sqrt{(a^2-b^2)}} = \frac{2}{\sqrt{1-(b/a)^2}}$$

$$= \frac{2}{\sqrt{(1 - \tan^2 x)}} = \frac{2 \cos x}{\sqrt{(\cos^2 x - \sin^2 x)}} \\ = \frac{2 \cos x}{\sqrt{(\cos 2x)}}$$

$$68. (3) \tan y = \frac{1 + \sqrt{1-x}}{1 + \sqrt{1+x}}$$

Let $x = \cos \theta$, then

$$\sqrt{1-x} = \sqrt{2} \sin(\theta/2); \sqrt{1+x} = \sqrt{2} \cos(\theta/2)$$

$$\Rightarrow \tan y = \frac{\sqrt{2} \left[\frac{1}{\sqrt{2}} + \sin \frac{\theta}{2} \right]}{\sqrt{2} \left[\frac{1}{\sqrt{2}} + \cos \frac{\theta}{2} \right]} = \frac{\sin \frac{\pi}{4} + \sin \frac{\theta}{2}}{\cos \frac{\pi}{4} + \cos \frac{\theta}{2}} \\ = \frac{2 \sin \left(\frac{\pi}{8} + \frac{\theta}{4} \right) \cos \left(\frac{\pi}{8} - \frac{\theta}{4} \right)}{2 \cos \left(\frac{\pi}{8} + \frac{\theta}{4} \right) \cos \left(\frac{\pi}{8} - \frac{\theta}{4} \right)} \\ = \tan \left(\frac{\pi}{8} + \frac{\theta}{4} \right)$$

$$\Rightarrow 4y = \frac{\pi}{2} + \theta$$

$$\text{or } \sin 4y = \cos \theta = x$$

$$69. (2) \frac{\cos 2B}{1} = \frac{\cos(A+C)}{\cos(A-C)}$$

Applying componendo and dividendo, we get

$$\frac{1 - \cos 2B}{1 + \cos 2B} = \frac{\cos(A-C) - \cos(A+C)}{\cos(A-C) + \cos(A+C)}$$

$$\text{or } \frac{2 \sin^2 B}{2 \cos^2 B} = \frac{2 \sin A \sin C}{2 \cos A \cos C}$$

$$\text{or } \tan^2 B = \tan A \tan C$$

Thus, $\tan A$, $\tan B$, $\tan C$ are in G.P.

$$70. (2) \frac{\cos(x-y)}{\cos(x+y)} + \frac{\cos(z+t)}{\cos(z-t)} = 0$$

$$\text{or } \frac{1 + \tan x \tan y}{1 - \tan x \tan y} + \frac{1 - \tan z \tan t}{1 + \tan z \tan t} = 0$$

$$\text{or } 1 + \tan z \tan t + \tan x \tan y + \tan x \tan y \tan z \tan t \\ + 1 - \tan z \tan t - \tan x \tan y + \tan x \tan y \tan z \tan t = 0$$

$$\text{or } \tan x \tan y \tan z \tan t = -1$$

$$71. (1) \tan \beta = 2 \sin \alpha \sin \gamma \operatorname{cosec}(\alpha + \gamma) = \frac{2 \sin \alpha \sin \gamma}{\sin(\alpha + \gamma)}$$

$$\text{or } \cot \beta = \frac{\sin(\alpha + \gamma)}{2 \sin \alpha \sin \gamma}$$

$$\text{or } 2 \cot \beta = \frac{\sin \alpha \cos \gamma + \cos \alpha \sin \gamma}{\sin \alpha \sin \gamma} = \cot \alpha + \cot \gamma$$

$$\text{or } \cot \alpha, \cot \beta, \cot \gamma \text{ are in A.P.}$$

$$72. (3) \text{ We have } \tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ \\ = (\tan 9^\circ + \tan 81^\circ) - (\tan 27^\circ + \tan 63^\circ) \\ = \frac{1}{\cos 9^\circ \cos 81^\circ} - \frac{1}{\cos 27^\circ \cos 63^\circ} \\ = \frac{1}{\sin 9^\circ \cos 9^\circ} - \frac{1}{\sin 27^\circ \cos 27^\circ}$$

$$= \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} \\ = 2 \left[\frac{\sin 54^\circ - \sin 18^\circ}{\sin 54^\circ \sin 18^\circ} \right] \\ = 2 \left[\frac{2 \cos 36^\circ \sin 18^\circ}{\sin 18^\circ \cos 36^\circ} \right] = 4$$

$$73. (2) \cos^3 x \sin 2x = \cos^2 x \cos x \sin 2x \\ = \left(\frac{1 + \cos 2x}{2} \right) \left(\frac{2 \sin 2x \cos x}{2} \right)$$

$$= \frac{1}{4} (1 + \cos 2x) (\sin 3x + \sin x)$$

$$= \frac{1}{4} \left[\sin 3x + \sin x + \frac{1}{2} (2 \sin 3x \cos 2x) \right]$$

$$+ \frac{1}{2} (2 \cos 2x \sin x) \\ = \frac{1}{4} \left[\sin 3x + \sin x + \frac{1}{2} (\sin 5x + \sin x) + \frac{1}{2} (\sin 3x - \sin x) \right] \\ = \frac{1}{4} \left[\sin x + \frac{3}{2} \sin 3x + \frac{1}{2} \sin 5x \right]$$

$$\Rightarrow a_1 = 1/4, a_3 = 3/8, n = 5$$

$$74. (3) \text{ Since } \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

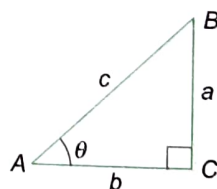
Putting $\theta = \frac{\pi}{9}$, we get

$$\tan \frac{\pi}{3} = \frac{3 \tan \frac{\pi}{9} - \tan^3 \frac{\pi}{9}}{1 - 3 \tan^2 \frac{\pi}{9}}$$

$$\text{or } 3 \left(1 - 3 \tan^2 \frac{\pi}{9} \right)^2 = \left(3 \tan \frac{\pi}{9} - \tan^3 \frac{\pi}{9} \right)^2$$

$$\text{or } \tan^6 \frac{\pi}{9} - 33 \tan^4 \frac{\pi}{9} + 27 \tan^2 \frac{\pi}{9} = 3$$

75. (4)



$$a = c \sin \theta, b = c \cos \theta$$

$$\Rightarrow \left(\frac{c}{a} + \frac{c}{b} \right)^2 = \left(\frac{1}{\sin \theta} + \frac{1}{\cos \theta} \right)^2 = \frac{4(1 + \sin 2\theta)}{\sin^2 2\theta} \\ = 4 \left(\frac{1}{\sin^2 2\theta} + \frac{1}{\sin 2\theta} \right), \text{ where } 0 < \theta < \frac{\pi}{2}$$

$$\Rightarrow \left(\frac{c}{a} + \frac{c}{b} \right)_{\min}^2 = 8, \text{ when } 2\theta = 90^\circ.$$

$$76. (2) a \operatorname{cosec} \alpha - b \sec \alpha = \frac{a}{\sin \alpha} - \frac{b}{\cos \alpha}$$

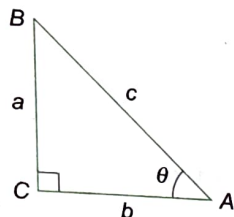
$$= \frac{\sqrt{a^2 + b^2}}{\sin \alpha \cos \alpha} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos \alpha - \frac{b}{\sqrt{a^2 + b^2}} \sin \alpha \right]$$

$$\text{Now, } \sin 3\alpha = \frac{a}{\sqrt{a^2 + b^2}} \text{ gives}$$

$$\sqrt{a^2 + b^2} \left[\frac{\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha} \right] = 2\sqrt{a^2 + b^2}$$

(1) We have
 $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$
 $= \tan 6^\circ \tan(60^\circ - 18^\circ) \tan(60^\circ + 6^\circ) \tan(60^\circ + 18^\circ)$
 $= \frac{[\tan 6^\circ \tan(60^\circ + 6^\circ) \tan 18^\circ \tan(60^\circ - 18^\circ) \tan(60^\circ + 18^\circ)]}{\tan 18^\circ}$
 $= \frac{\tan 6^\circ \tan(60^\circ + 6^\circ) \tan(3 \times 18^\circ)}{\tan 18^\circ}$
 $= \frac{\tan 6^\circ \tan(60^\circ - 6^\circ) \tan(60^\circ + 6^\circ)}{\tan 18^\circ}$
 $= \frac{\tan 18^\circ}{\tan 18^\circ} = 1$

(4) $\Delta = \frac{1}{2}ab$
 $\Rightarrow ab = 60$
 $b = c \cos \theta, a = c \sin \theta$
 $\Rightarrow \Delta = \frac{1}{2}c^2 \sin \theta \cos \theta = \frac{c^2 \sin 2\theta}{4}$
 $= 30$



or $c^2 = 120 \operatorname{cosec} 2\theta$

or $c_{\min}^2 = 120$

or $c_{\min} = 2\sqrt{30}$

(3) $\sqrt{2} \cos A = \cos B + \cos^3 B$... (i)

and $\sqrt{2} \sin A = \sin B - \sin^3 B$... (ii)

$\Rightarrow \sqrt{2} \sin A \cos B - \sqrt{2} \cos A \sin B$
 $= (\sin B - \sin^3 B) \cos B - (\cos B + \cos^3 B) \sin B$
 $= -\sin B \cos B$

$\Rightarrow \sin(A - B) = -\frac{1}{2\sqrt{2}} \sin 2B$

Now squaring and adding Eqs. (i) and (ii), we get

$2 = \cos^2 B + \sin^2 B + \cos^6 B + \sin^6 B + 2(\cos^4 B - \sin^4 B)$

$\Rightarrow 1 = (\cos^2 A + \sin^2 A)^3 - 3 \cos^2 A \sin^2 A (\cos^2 A + \sin^2 A) + 2 \cos 2B$

$\Rightarrow 1 = 1 - (3/4) \sin^2 2B + 2 \cos 2B$

$\Rightarrow -3 \sin^2 2B + 8 \cos 2B = 0$

$\Rightarrow 3 \cos^2 2B + 8 \cos 2B - 3 = 0$

$\Rightarrow \cos 2B = \frac{1}{3}$

$\Rightarrow \sin 2B = \pm \frac{2\sqrt{2}}{3} \therefore \sin(A - B) = \pm \frac{1}{3}$

(2) $P^2 \sec^2 \theta + P^2 \operatorname{cosec}^2 \theta = (2\sqrt{2})^2 P^2$

$\Rightarrow \frac{1}{\sin^2 \theta \cos^2 \theta} = 8$

$\Rightarrow \sin^2 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2$

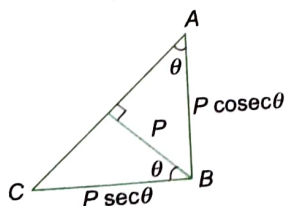
$\Rightarrow 2\theta = \pi/4 \text{ or } 3\pi/4$

$\Rightarrow \theta = \pi/8, 3\pi/8$

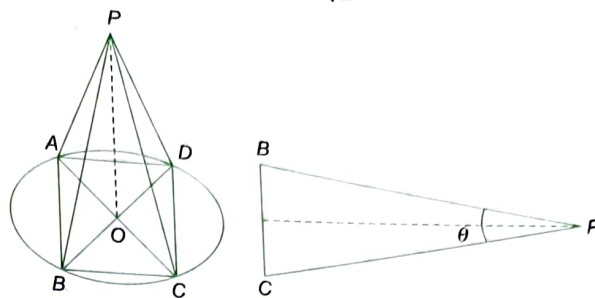
(3) $OP = 4$

$OA = OB = OC = OD = 3$

$AP = BP = CP = DP = \sqrt{3^2 + 4^2} = 5$



$ABCD$ is a square of side length $\frac{6}{\sqrt{2}} = 3\sqrt{2}$.



$\sin \frac{\theta}{2} = \frac{\frac{1}{2}BC}{BP} = \frac{\frac{3}{\sqrt{2}}}{5} = \frac{3}{5\sqrt{2}}$

$\therefore \cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = \frac{16}{25}$

82. (2) From figure,

$x \cos(\theta + 30^\circ) = d$... (i)

and $x \sin \theta = 1 - d$... (ii)

Dividing $\sqrt{3} \cot \theta = \frac{1+d}{1-d}$,

squaring Eq. (ii) and putting the value of $\cot \theta$, we get

$x^2 = \frac{1}{3} (4d^2 - 4d + 4)$

83. (3) Let $A = \sin^{-1} a$, $B = \sin^{-1} b$, and $C = \sin^{-1} c$.

We have $A + B + C = \pi$.

$a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$

$= \frac{1}{2} (\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2} [4 \sin A \sin B \sin C] = 2abc$

84. (4) $\cos 2A + \cos 2B + \cos 2C$

$= 2 \cos(A+B) \cos(A-B) + \cos 2C$

$= 2 \cos\left(\frac{3\pi}{2} - C\right) \cos(A-B) + \cos 2C$

$= -2 \sin C \cos(A-B) + 1 - 2 \sin^2 C$

$= 1 - 2 \sin C (\cos(A-B) + \sin C)$

$= 1 - 2 \sin C \{\cos(A-B) + \sin[3\pi/2 - (A+B)]\}$

$= 1 - 2 \sin C [\cos(A-B) - \cos(A+B)]$

$= 1 - 4 \sin A \sin B \sin C$

85. (1) We have $\frac{\sin 2\beta}{3 - \cos 2\beta} = \frac{2 \sin \beta \cdot \cos \beta}{2 - 2 \cos 2\beta + 1 + \cos 2\beta}$

$= \frac{2 \sin \beta \cdot \cos \beta}{4 \sin^2 \beta + 2 \cos^2 \beta}$

$= \frac{\tan \beta}{1 + 2 \tan^2 \beta}$

$= \frac{2 \tan \beta - \tan \beta}{1 + 2 \tan^2 \beta}$

$= \tan(\alpha - \beta)$

$$\therefore \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta} = \frac{2 \tan \beta - \tan \beta}{1 + 2 \tan^2 \beta}$$

$$\therefore \tan \alpha = 2 \tan \beta$$

$$\begin{aligned} 86. (2) \quad \sin^2 A - \sin^2 B + \sin^2 C &= \sin(A+B) \sin(A-B) + \sin^2 C \\ &= \sin C (\sin(A-B) + \sin C) \\ &= \sin C (\sin(A-B) + \sin(A+B)) \\ &= 2 \sin A \cos B \sin C \end{aligned}$$

$$\begin{aligned} 87. (4) \quad \sum_{r=0}^{10} \cos^3 \frac{r\pi}{3} &= \frac{1}{4} \sum_{r=0}^{10} \left(3 \cos \frac{r\pi}{3} + \cos r\pi \right) \\ &= \frac{1}{4} \left[3 \left(\cos 0 + \cos \frac{\pi}{3} + \dots + \cos \frac{10\pi}{3} \right) + (1 - 1 + \dots - 1 + 1) \right] \\ &= \frac{3}{4} \left[\frac{\cos \left(\frac{10\pi}{6} \right) \sin \left(\frac{11\pi}{6} \right)}{\sin \frac{\pi}{6}} \right] + \frac{1}{4} = -\frac{1}{8} \end{aligned}$$

$$\begin{aligned} 88. (3) \quad \text{Denominator} &= \sin A + \sin B - \sin C \\ &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2} \\ &= 2 \cos \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \sin \frac{C}{2} \right] \\ &= 2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] \\ &= 2 \cos \frac{C}{2} \left[2 \sin \frac{A}{2} \sin \frac{B}{2} \right] \\ &= 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \end{aligned}$$

$$\text{Also } \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\Rightarrow \frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = \cot \frac{A}{2} \cot \frac{B}{2}$$

$$\begin{aligned} 89. (1) \quad \frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} &= \frac{4 \sin A \sin B \sin C}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \\ &= 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \quad \left[\because \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \right] \end{aligned}$$

$$\begin{aligned} 90. (3) \quad \text{We have } \cos^2 A + \cos^2 B - (1 - \cos^2 C) &= 0 \\ \text{or } \cos^2 A + \cos^2 B - \sin^2 C &= 0 \\ \text{or } \cos^2 A + \cos(B+C) \cos(B-C) &= 0 \\ \text{or } \cos A [\cos A - \cos(B-C)] &= 0 \\ \text{or } \cos A [\cos(B+C) + \cos(B-C)] &= 0 \\ \text{or } 2 \cos A \cos B \cos C &= 0 \end{aligned}$$

Hence, either A or B or C is 90° .

91. (1) In a triangle,

$$\begin{aligned} \tan A + \tan B + \tan C &= \tan A \tan B \tan C \quad \dots(i) \\ \Rightarrow 6 &= 2 \tan C \\ \text{or } \tan C &= 3 \end{aligned}$$

$$\text{Also } \tan A + \tan B = 6 - 3 = 3 \quad \dots(ii)$$

By Eqs. (i) and (ii), $\tan A$ and $\tan B$ are roots of $x^2 - 3x + 2 = 0$.
Thus,

$$\tan A, \tan B = 2, 1 \text{ or } 1, 2 \text{ and } \tan C = 3$$

$$92. (3) \quad \cos x + \cos y - \cos(x+y) = \frac{3}{2}$$

$$\text{or } 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) - 2 \cos^2 \left(\frac{x+y}{2} \right) + 1 = \frac{3}{2}$$

$$\text{or } 2 \cos^2 \left(\frac{x+y}{2} \right) - 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) + \frac{1}{2} = 0$$

Now $\cos \left(\frac{x+y}{2} \right)$ is always real, then discriminant ≥ 0 . Thus,

$$4 \cos^2 \left(\frac{x-y}{2} \right) - 4 \geq 0$$

$$\text{or } \cos^2 \left(\frac{x-y}{2} \right) \geq 1$$

$$\text{or } \cos^2 \left(\frac{x-y}{2} \right) = 1$$

$$\text{or } \frac{x-y}{2} = 0 \text{ or } x = y$$

$$93. (1) \quad a \sin x + b \cos(x+\theta) + b \cos(x-\theta) = d$$

$$\Rightarrow a \sin x + 2b \cos x \cos \theta = d$$

$$\Rightarrow |d| \leq \sqrt{a^2 + 4b^2 \cos^2 \theta}$$

$$\Rightarrow \frac{d^2 - a^2}{4b^2} \leq \cos^2 \theta$$

$$\Rightarrow |\cos \theta| \geq \frac{\sqrt{d^2 - a^2}}{2|b|}$$

$$94. (4) \quad u^2 = a^2 + b^2 + 2\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \times \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$$

$$= a^2 + b^2 + 2\sqrt{\sin^2 \theta \cos^2 \theta (a^4 + b^4) + a^2 b^2 (\sin^4 \theta + \cos^4 \theta)}$$

$$= a^2 + b^2 + 2\sqrt{a^2 b^2 (1 - 2 \sin^2 \theta \cos^2 \theta) + (a^4 + b^4) \sin^2 \theta \cos^2 \theta}$$

$$= (a^2 + b^2) + 2\sqrt{a^2 b^2 + (a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta}$$

$$= (a^2 + b^2) + 2\sqrt{a^2 b^2 + \frac{(a^2 - b^2)^2}{4} \sin^2 2\theta}$$

$$\text{Max. } u^2 = (a^2 + b^2) + 2\sqrt{a^2 b^2 + \frac{(a^2 - b^2)^2}{4}}$$

$$\text{Min. } u^2 = (a^2 + b^2) + 2ab$$

$$\begin{aligned} \Rightarrow \text{Difference} &= 2\sqrt{a^2 b^2 + \frac{(a^2 - b^2)^2}{4}} - 2ab \\ &= \sqrt{4a^2 b^2 + a^4 + b^4 - 2a^2 b^2} - 2ab \\ &= \sqrt{(a^2 + b^2)^2} - 2ab \\ &= a^2 + b^2 - 2ab \\ &= (a - b)^2 \end{aligned}$$

$$95. (4) \quad \tan x = n \tan y, \cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\begin{aligned} \Rightarrow \cos(x-y) &= \cos x \cos y (1 + \tan x \tan y) \\ &= \cos x \cos y (1 + n \tan^2 y) \end{aligned}$$

$$\Rightarrow \sec^2(x-y) = \frac{\sec^2 x \sec^2 y}{(1 + n \tan^2 y)^2}$$

$$\begin{aligned}
 &= \frac{(1 + \tan^2 x)(1 + \tan^2 y)}{(1 + n \tan^2 y)^2} \\
 &= \frac{(1 + n^2 \tan^2 y)(1 + \tan^2 y)}{(1 + n \tan^2 y)^2} \\
 &= 1 + \frac{(n-1)^2 \tan^2 y}{(1 + n \tan^2 y)^2}
 \end{aligned}$$

Now, $\left(\frac{1 + n \tan^2 y}{2}\right)^2 \geq n \tan^2 y$ ($\because$ A.M. $\geq$ G.M.)

or $\frac{\tan^2 y}{(1 + n \tan^2 y)^2} \leq \frac{1}{4n}$

$\Rightarrow \sec^2(x-y) \leq 1 + \frac{(n-1)^2}{4n} = \frac{(n+1)^2}{4n}$

(2) If $a, b > 0$
Using A.M. $\geq$ G.M., we get

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$$

$$\Rightarrow f(x) \geq \frac{2}{\sqrt{\cos\left(\frac{\pi}{6} - x\right) \cos\left(\frac{\pi}{6} + x\right)}}$$

$$= \frac{2}{\sqrt{\cos^2 \frac{\pi}{6} - \sin^2 x}}$$

$$= \frac{2}{\sqrt{\frac{3}{4} - \frac{1 - \cos 2x}{2}}}$$

$$= \frac{2}{\sqrt{\frac{1}{4} + \frac{\cos 2x}{2}}}$$

Now for $0 \leq x \leq \frac{\pi}{3}$, $-\frac{1}{2} \leq \cos 2x \leq 1$

$$\Rightarrow 0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2}$$

$$\Rightarrow f(x) \geq \frac{4}{\sqrt{3}}$$

Since 'f' is continuous range of 'f' is $\left[\frac{4}{\sqrt{3}}, \infty\right)$.

(4) Let $y = \cos x \left\{ \sin x + \sqrt{\sin^2 x + \sin^2 \frac{\pi}{6}} \right\}$ is

$$\Rightarrow y^2 (1 + \tan^2 x) - 2y \tan x = \frac{1}{4}$$

$$\Rightarrow y^2 \tan^2 x - 2y \tan x + \left(y^2 - \frac{1}{4}\right) \geq 0$$

For real $\tan x$,
disc. ≥ 0

$$4y^2 - 4y^2 \left(y^2 - \frac{1}{4}\right) \geq 0$$

$$\Rightarrow 0 \leq y^2 \leq \frac{5}{4}$$

$\therefore$ Maximum value of $y = \frac{\sqrt{5}}{2}$

98. (2) From the third relation we get

$$\cos \theta \cos \phi + \sin \theta \sin \phi = \sin \beta \sin \gamma$$

$$\Rightarrow \sin^2 \theta \sin^2 \phi = (\cos \theta \cos \phi - \sin \beta \sin \gamma)^2$$

$$\Rightarrow \left(1 - \frac{\sin^2 \beta}{\sin^2 \alpha}\right) \left(1 - \frac{\sin^2 \gamma}{\sin^2 \alpha}\right) = \left(\frac{\sin \beta \sin \gamma}{\sin^2 \alpha} - \sin \beta \sin \gamma\right)^2$$

$$\Rightarrow (\sin^2 \alpha - \sin^2 \beta)(\sin^2 \alpha - \sin^2 \gamma) = \sin^2 \beta \sin^2 \gamma (1 - \sin^2 \alpha)^2$$

$$\Rightarrow \sin^4 \alpha (1 - \sin^2 \beta \sin^2 \gamma) - \sin^2 \alpha (\sin^2 \beta + \sin^2 \gamma - 2 \sin^2 \beta \sin^2 \gamma) = 0$$

$$\therefore \sin^2 \alpha = \frac{\sin^2 \beta + \sin^2 \gamma - 2 \sin^2 \beta \sin^2 \gamma}{1 - \sin^2 \beta \sin^2 \gamma}$$

$$\text{and } \cos^2 \alpha = \frac{1 - \sin^2 \beta - \sin^2 \gamma + \sin^2 \beta \sin^2 \gamma}{1 - \sin^2 \beta \sin^2 \gamma}$$

$$\Rightarrow \tan^2 \alpha = \frac{\sin^2 \beta - \sin^2 \beta \sin^2 \gamma + \sin^2 \gamma - \sin^2 \beta \sin^2 \gamma}{\cos^2 \beta - \sin^2 \gamma (1 - \sin^2 \beta)}$$

$$= \frac{\sin^2 \beta \cos^2 \gamma + \cos^2 \beta \sin^2 \gamma}{\cos^2 \beta \cos^2 \gamma}$$

$$= \tan^2 \beta + \tan^2 \gamma$$

$$\Rightarrow \tan^2 \alpha - \tan^2 \beta - \tan^2 \gamma = 0$$

$$99. (1) S_n = \sum_{r=1}^n \frac{\tan\left(\frac{\theta}{2^r}\right)}{2^{r-1} \cos\left(\frac{\theta}{2^{r-1}}\right)}$$

$$= \sum_{r=1}^n \frac{\sin\left(\frac{\theta}{2^r}\right)}{2^{r-1} \cos\left(\frac{\theta}{2^r}\right) \cos\left(\frac{\theta}{2^{r-1}}\right)}$$

$$= \sum_{r=1}^n \frac{2 \sin^2\left(\frac{\theta}{2^r}\right)}{2^{r-1} \left\{ 2 \sin\left(\frac{\theta}{2^r}\right) \cos\left(\frac{\theta}{2^r}\right) \right\} \cos\left(\frac{\theta}{2^{r-1}}\right)}$$

$$= \sum_{r=1}^n \frac{1 - \cos\left(\frac{\theta}{2^{r-1}}\right)}{2^{r-1} \sin\left(\frac{\theta}{2^{r-1}}\right) \cos\left(\frac{\theta}{2^{r-1}}\right)}$$

$$= \sum_{r=1}^n \left(\frac{1}{2^{r-2} \sin\left(\frac{\theta}{2^{r-2}}\right)} - \frac{1}{2^{r-1} \sin\left(\frac{\theta}{2^{r-1}}\right)} \right)$$

$$= \frac{2}{\sin 2\theta} - \frac{1}{2^{n-1} \sin\left(\frac{\theta}{2^{n-1}}\right)}$$

$$\therefore S_\infty = \frac{2}{\sin 2\theta} - \lim_{n \rightarrow \infty} \frac{\frac{\theta}{2^{n-1}} \cdot \frac{1}{\theta}}{\sin\left(\frac{\theta}{2^{n-1}}\right)} = \frac{2}{\sin \theta} - \frac{1}{\theta}$$

$$100. (2) x \sin a + y \sin 2a + z \sin 3a = \sin 4a$$

$$\Rightarrow x \sin a + y \times 2 \sin a \cos a + z \times \sin a (3 - 4 \sin^2 a) = 2 \times 2 \sin a \cos a \cos 2a$$

$$\Rightarrow x + 2y \cos a + z (3 + 4 \cos^2 a - 4) = 4 \cos a (2 \cos^2 a - 1)$$

[as $\sin a \neq 0$]

5.52 Trigonometry

$$\Rightarrow 8 \cos^3 a - 4z \cos^2 a - (2y + 4) \cos a + (z - x) = 0$$

$$\Rightarrow \cos^3 a - \left(\frac{z}{2}\right) \cos^2 a - \left(\frac{y+2}{4}\right) \cos a + \left(\frac{z-x}{8}\right) = 0$$

Which shows that $\cos a$ is root of the equation

$$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$$

Similarly, from second and third equations, we can show that $\cos b$ and $\cos c$ are the roots of the given equation.

Multiple Correct Answers Type

1. (1), (2)

$$2 \sin \alpha \cos \alpha = 2 \cos^2 \beta$$

$$\sin 2\alpha = 1 + \cos 2\beta$$

$$\therefore \cos 2\beta = -(1 - \sin 2\alpha)$$

$$= -\left(1 - \cos\left(\frac{\pi}{2} - 2\alpha\right)\right)$$

$$= -2 \sin^2\left(\frac{\pi}{4} - \alpha\right)$$

$$= -2 \cos^2\left(\frac{\pi}{4} + \alpha\right)$$

2. (1), (2), (3)

$$(1) \tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta}. \text{ Hence, option (1) is correct.}$$

$$(2) \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Hence, option (2) is correct.

$$(3) \tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}. \text{ Hence, option (3) is correct.}$$

$$(4) \sin \theta = 1/3, \text{ which is rational but } \cos 3\theta = \cos \theta (4 \cos^2 \theta - 3), \text{ which is irrational.}$$

Hence, option (4) is incorrect.

3. (1), (2), (3), (4)

$$(1) \sin\left(\frac{11\pi}{12}\right) \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{12}\right) \cos\left(\frac{\pi}{12}\right)$$

$$= \frac{1}{2} \sin\left(\frac{\pi}{6}\right) = \frac{1}{4} \in \mathbb{Q}$$

$$(2) \operatorname{cosec}\left(\frac{9\pi}{10}\right) \sec\left(\frac{4\pi}{5}\right) = -\operatorname{cosec}\left(\frac{\pi}{10}\right) \sec\left(\frac{\pi}{5}\right)$$

$$= \frac{1}{-\sin 18^\circ \cos 36^\circ}$$

$$= -\frac{16}{(\sqrt{5}-1)(\sqrt{5}+1)}$$

$$= -4 \in \mathbb{Q}$$

$$(3) \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) = 1 - 2 \sin^2\left(\frac{\pi}{8}\right) \cos^2\left(\frac{\pi}{8}\right)$$

$$= 1 - \frac{1}{2} \sin^2\left(\frac{\pi}{4}\right)$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \in \mathbb{Q}$$

$$(4) 2 \cos^2 \frac{\pi}{9} \cdot 2 \cos^2 \frac{2\pi}{9} \cdot 2 \cos^2 \frac{4\pi}{9}$$

$$= 8 (\cos 20^\circ \cos 40^\circ \cos 80^\circ)^2 = \frac{1}{8} \in \mathbb{Q}$$

4. (1), (2), (4)

$$(\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x) > \frac{5}{8}$$

$$\text{or } 1 - 3 \sin^2 x \cos^2 x > \frac{5}{8}$$

$$\text{or } \frac{3}{8} > 3 \sin^2 x \cos^2 x$$

$$\Rightarrow 1 - 2 \sin^2 2x > 0$$

$$\Rightarrow \cos 4x > 0$$

$$\Rightarrow 4x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow 4x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right),$$

$n \in \mathbb{Z}$, generalizing now verify.

5. (1), (3)

We have

$$f(x) = x^2 - 2\sqrt{(\sin \sqrt{3} - \sin \sqrt{2})} x - (\cos \sqrt{3} - \cos \sqrt{2})$$

$$\therefore D = 4 \left[(\sin \sqrt{3} - \sin \sqrt{2}) + (\cos \sqrt{3} - \cos \sqrt{2}) \right]$$

$$= 8\sqrt{2} \sin\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right) \cos\left(\frac{\pi}{4} + \frac{\sqrt{2}+\sqrt{3}}{2}\right)$$

$$\text{As } \sin\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right) > 0 \text{ and } \frac{\pi}{2} < \frac{\sqrt{2}+\sqrt{3}}{2} < \pi$$

$$\therefore D_1 < 0$$

Hence, $f(x) > 0 \forall x \in \mathbb{R}$.

6. (1), (2), (3)

$$(1) \tan \alpha \tan 2\alpha \tan 3\alpha = \tan 3\alpha - \tan 2\alpha - \tan \alpha$$

always holds good. ($\because \tan 2\alpha = \tan(3\alpha - \alpha)$)

$$(2) \text{R.H.S.} = \frac{\sin 4\alpha + \sin 2\alpha}{\sin 2\alpha \sin 4\alpha}$$

$$= \frac{2 \sin 3\alpha \cos \alpha}{\sin 2\alpha \sin 4\alpha} = \frac{2 \sin 4\alpha \cdot \cos \alpha}{\sin 2\alpha \cdot \sin 4\alpha}$$

$$= \frac{1}{\sin \alpha} = \operatorname{cosec} \alpha \text{ (using } \pi/7 = \alpha)$$

Hence, (2) is correct.

$$\text{Also } \cos 2\alpha = \cos \frac{2\pi}{7} = -\cos\left(\pi - \frac{5\pi}{7}\right)$$

$$= -\cos\left(\frac{5\pi}{7}\right) = -\cos 5\alpha$$

$$(3) \cos \alpha + \cos 3\alpha + \cos 5\alpha = \frac{\sin 3\alpha}{\sin \alpha} \cos 3\alpha$$

$$= \frac{\sin 6\alpha}{2\sin \alpha} = \frac{\sin \frac{6\pi}{7}}{2\sin \frac{\pi}{7}}$$

$$= \frac{\sin\left(\pi - \frac{\pi}{7}\right)}{2\sin \frac{\pi}{7}} = \frac{1}{2}$$

$$(4) 8 \cos \alpha \cos 2\alpha \cos 4\alpha = \frac{\sin 8\alpha}{\sin \alpha} = \frac{\sin \frac{8\pi}{7}}{\sin \frac{\pi}{7}} = -\frac{\sin \frac{\pi}{7}}{\sin \frac{\pi}{7}} = -1$$

7. (1), (3)

$$(1) \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha} = 2 \cot 2\alpha$$

$$(2) \frac{1+t}{1-t} - \frac{1-t}{1+t}, \text{ where}$$

$$t = \tan \alpha = \frac{(1+t)^2 - (1-t)^2}{1-t^2}$$

$$= \frac{4t}{1-t^2} = \frac{2 \times 2 \tan \alpha}{1 - \tan^2 \alpha} = 2 \tan 2\alpha$$

Hence, (2) is incorrect.

$$(3) \frac{1+t}{1-t} + \frac{1-t}{1+t} = \frac{(1+t)^2 + (1-t)^2}{1-t^2}$$

$$= \frac{2(1+t^2)}{1-t^2} \quad (\text{where } t = \tan \alpha)$$

$$= \frac{2}{\cos 2\alpha} = 2 \sec 2\alpha$$

Hence, (3) is correct.

$$(4) \tan \alpha + \cot \alpha = \frac{1}{\cos \alpha \sin \alpha} = \frac{2}{\sin 2\alpha} = 2 \operatorname{cosec} 2\alpha$$

Hence, (4) is incorrect.

8. (1), (4)

$$y = \frac{(1 + \tan^2 x)^2}{1 + \tan^2 x} = 1 + \tan^2 x$$

$$= 1 + (2 - \sqrt{3})^2$$

$$= 8 - 4\sqrt{3} = 4(2 - \sqrt{3})$$

$$= 4 \left[\left(\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} \right)^2 \right]$$

$$= 4 \left[4 \left(\frac{\sqrt{3} - 1}{2\sqrt{2}} \right)^2 \right]$$

$$= 16 \sin^2 \frac{\pi}{12}$$

9. (2), (4)

$$\tan(\alpha + \beta) = \frac{15}{8}$$

$$\text{and } \operatorname{cosec} \gamma = \frac{17}{8}$$

$$\Rightarrow \tan \gamma = \frac{8}{15}$$

$$\therefore \alpha + \beta + \gamma = \frac{\pi}{2}$$

Hence, (2) is true.

$$\text{Also } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\text{or } \cot \gamma = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\text{or } \tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1$$

10. (1), (2), (3)

$$\text{We have } f_n(\theta) = \cot\left((3n-1)\frac{\theta}{4}\right)$$

$$\therefore f_3\left(\frac{3\pi}{16}\right) = \cot\left(8 \times \frac{1}{4} \times \frac{3\pi}{16}\right) = \cot \frac{3\pi}{8} = \sqrt{2} - 1$$

$$f_5\left(\frac{\pi}{28}\right) = \cot\left(14 \times \frac{1}{4} \times \frac{\pi}{28}\right) = \cot \frac{\pi}{8} = \sqrt{2} + 1$$

$$f_7\left(\frac{\pi}{60}\right) = \cot\left(20 \times \frac{1}{4} \times \frac{\pi}{60}\right) = \cot \frac{\pi}{12} = (2 + \sqrt{3})$$

11. (1), (3), (4)

$$\sin x \cos 20^\circ + \cos x \sin 20^\circ = 2 \sin x \cos 40^\circ$$

$$\text{or } \sin 20^\circ \cos x = \sin x (2 \cos 40^\circ - \cos 20^\circ)$$

$$\text{or } \tan x = \frac{\sin 20^\circ}{2 \cos 40^\circ - \cos 20^\circ}$$

$$= \frac{\sin 20^\circ}{\cos 40^\circ + \cos 40^\circ - \cos 20^\circ}$$

$$= \frac{\sin 20^\circ}{\cos 40^\circ + 2 \sin 30^\circ \sin(-10^\circ)}$$

$$= \frac{\sin 20^\circ}{\sin 50^\circ - \sin 10^\circ} = \frac{\sin 20^\circ}{2 \cos 30^\circ \sin 20^\circ}$$

$$= \frac{1}{\sqrt{3}} \text{ or } x = 30^\circ$$

12. (1), (2), (3)

$$\cos^2(\alpha + \beta) + \cos^2(\alpha - \beta) - \cos 2\alpha \cos 2\beta$$

$$= \cos^2(\alpha + \beta) + 1 - \sin^2(\alpha - \beta) - \cos 2\alpha \cos 2\beta$$

$$= 1 + \cos 2\alpha \cos 2\beta - \cos 2\alpha \cos 2\beta$$

$$= 1$$

13. (1), (3)

$$\cot^3 \alpha + \cot^2 \alpha + \cot \alpha = 1$$

$$\Rightarrow \tan^3 \alpha - \tan^2 \alpha - \tan \alpha - 1 = 0$$

$$\Rightarrow \tan \alpha (\tan^2 \alpha - 1) = 1 + \tan^2 \alpha$$

$$\Rightarrow \tan \alpha = -\left(\frac{1 + \tan^2 \alpha}{1 - \tan^2 \alpha}\right)$$

$$\Rightarrow \cos 2\alpha \cdot \tan \alpha = -1$$

$$\text{Now } \cos 2\alpha - \tan 2\alpha = -\frac{1}{\tan \alpha} - \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$= -\left(\frac{1 - \tan^2 \alpha + 2 \tan^2 \alpha}{\tan \alpha (1 - \tan^2 \alpha)}\right)$$

$$= -\frac{1 + \tan^2 \alpha}{\tan \alpha (1 - \tan^2 \alpha)}$$

$$= -\frac{1}{\tan \alpha \cos 2\alpha}$$

$$= 1$$

14. (2), (4)

$$\begin{aligned} 2p &= \cos(A - B - C + D) - \cos(A - B + C - D) \\ \Rightarrow 2q &= \cos(B - C - A + D) - \cos(B - C + A - D) \\ \Rightarrow 2r &= \cos(C - A - B + D) - \cos(C - A + B - D) \\ \Rightarrow 2p + 2q + 2r &= 0 \\ p + q + r &= 0 \text{ and } p^3 + q^3 + r^3 = 3pqr \end{aligned}$$

15. (1), (2)

$$\begin{aligned} \cos x - \sin \alpha \cot \beta \sin x &= \cos \alpha \\ \Rightarrow \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} - \sin \alpha \cot \beta \frac{2 \tan(x/2)}{1 + \tan^2(x/2)} &= \cos \alpha \\ \Rightarrow \tan^2 \frac{x}{2} (1 + \cos \alpha) + 2 \sin \alpha \cot \beta \tan \frac{x}{2} - (1 - \cos \alpha) &= 0 \\ \Rightarrow \tan^2 \frac{x}{2} + \frac{2 \sin \alpha \cot \beta}{1 + \cos \alpha} \tan \frac{x}{2} - \frac{1 - \cos \alpha}{1 + \cos \alpha} &= 0 \\ \Rightarrow \tan^2 \frac{x}{2} + 2 \tan \frac{\alpha}{2} \cot \beta \tan \frac{x}{2} - \tan^2 \frac{\alpha}{2} &= 0 \\ \Rightarrow \tan^2 \frac{x}{2} + 2 \tan \frac{\alpha}{2} \cdot \frac{1}{2} \left(\cot \frac{\beta}{2} - \tan \frac{\beta}{2} \right) & \\ \tan \frac{x}{2} - \tan^2 \frac{\alpha}{2} &= 0 \\ \Rightarrow \left(\tan \frac{x}{2} + \cot \frac{\beta}{2} \tan \frac{\alpha}{2} \right) \left(\tan \frac{x}{2} - \tan \frac{\beta}{2} \tan \frac{\alpha}{2} \right) &= 0 \\ \Rightarrow \tan \left(\frac{x}{2} \right) = -\tan \left(\frac{\alpha}{2} \right) \cot \left(\frac{\beta}{2} \right) & \\ \text{or } \tan \left(\frac{x}{2} \right) = \tan \left(\frac{\alpha}{2} \right) \tan \left(\frac{\beta}{2} \right) & \end{aligned}$$

16. (1), (2), (3)

$$\begin{aligned} f(x) &= ab \sin x + b\sqrt{1-a^2} \cos x + c, \text{ where } |a| < 1, b < 0 \\ f(x) &= \sqrt{a^2 b^2 + b^2 - b^2 a^2} \sin(x + \alpha) + c \\ &= b \sin(x + \alpha) + c, \text{ where } \tan \alpha = \frac{b\sqrt{1-a^2}}{ab} = \frac{\sqrt{1-a^2}}{a} \\ &= b \cos(x - \alpha) + c, \text{ where } \tan \alpha = \frac{ab}{b\sqrt{1-a^2}} = \frac{a}{\sqrt{1-a^2}} \\ f(x)_{\max} - f(x)_{\min} &= c + b - (c - b) = 2b \\ f(x) &= c \text{ if } x + \alpha = 0 \\ \text{or } x &= -\alpha \\ \text{or } x &= -\cos^{-1} a \end{aligned}$$

17. (1), (2), (3), (4)

$$\begin{aligned} P(k) &= \left(1 + \cos \frac{\pi}{4k} \right) \left(1 + \cos \left(\frac{\pi}{2} - \frac{\pi}{4k} \right) \right) \left(1 + \cos \left(\frac{\pi}{2} + \frac{\pi}{4k} \right) \right) \\ & \quad \left(1 + \cos \left(\pi - \frac{\pi}{4k} \right) \right) \\ &= \left(1 + \cos \frac{\pi}{4k} \right) \left(1 + \sin \frac{\pi}{4k} \right) \left(1 - \sin \frac{\pi}{4k} \right) \left(1 - \cos \frac{\pi}{4k} \right) \\ &= \left(1 - \cos^2 \frac{\pi}{4k} \right) \left(1 - \sin^2 \frac{\pi}{4k} \right) \\ &= \frac{4 \sin^2 \frac{\pi}{4k} \cdot \cos^2 \frac{\pi}{4k}}{4} \end{aligned}$$

$$P(k) = \frac{1}{4} \sin^2 \left(\frac{\pi}{2k} \right)$$

$$\Rightarrow P(3) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

$$\Rightarrow P(4) = \frac{1}{4} \sin^2 \frac{\pi}{2k} = \frac{1}{4} \sin^2 \frac{\pi}{8} = \frac{1}{8} \left(1 - \cos \frac{\pi}{4} \right) = \frac{2 - \sqrt{2}}{16}$$

$$\Rightarrow P(5) = \frac{1}{4} \sin^2 \frac{\pi}{10} = \frac{1}{8} \left(2 \sin^2 \frac{\pi}{10} \right) = \frac{1}{8} (1 - \cos 36^\circ) \\ = \frac{1}{8} \left(1 - \frac{\sqrt{5} + 1}{4} \right) = \frac{3 - \sqrt{5}}{32}$$

$$\Rightarrow P(6) = \frac{1}{4} \sin^2 \frac{\pi}{12} = \frac{1}{8} \left(2 \sin^2 \frac{\pi}{12} \right) = \frac{1}{8} \left(1 - \cos \frac{\pi}{6} \right) \\ = \frac{1}{8} \left(1 - \frac{\sqrt{3}}{2} \right) = \frac{2 - \sqrt{3}}{16}$$

18. (1), (2), (4)

$$\begin{aligned} 3 \sin \beta &= \sin(2\alpha + \beta), \\ \tan(\alpha + \beta) - 2 \tan \alpha & \\ &= \tan(\alpha + \beta) - \tan \alpha - \tan \alpha \\ &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} - \frac{\sin \alpha}{\cos \alpha} - \tan \alpha \\ &= \frac{\sin(\alpha + \beta - \alpha)}{\cos \alpha \cos(\alpha + \beta)} - \tan \alpha \\ &= \frac{\sin \beta}{\cos(\alpha + \beta) \cos \alpha} - \frac{\sin \alpha}{\cos \alpha} \\ &= \frac{\sin \beta - \sin \alpha \cos(\alpha + \beta)}{\cos(\alpha + \beta) \cos \alpha} \\ &= \frac{2 \sin \beta - [\sin(2\alpha + \beta) - \sin \beta]}{2 \cos(\alpha + \beta) \cos \alpha} \\ &= \frac{-[\sin(2\alpha + \beta) - 3 \sin \beta]}{2 \cos(\alpha + \beta) \cos \alpha} = 0 \end{aligned}$$

(Given)

19. (1), (2), (3), (4)

$$\begin{aligned} x &= \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} + \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \\ \Rightarrow x^2 &= a^2 + b^2 + \\ & \quad 2 \sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)} \\ &= a^2 + b^2 + 2k, \end{aligned}$$

$$\text{where } k = \sqrt{[(a^2 + b^2) - (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)] \times (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)}$$

$$\therefore x = a^2 + b^2 + 2 \sqrt{(a^2 + b^2) p - p^2}$$

$$\begin{aligned} \text{where } p &= a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \\ &= \frac{a^2}{2} (1 - \cos 2\alpha) + \frac{b^2}{2} (1 + \cos 2\alpha) \\ &= \frac{1}{2} [a^2 + b^2 - (a^2 - b^2) \cos 2\alpha] \end{aligned}$$

$$\text{Also, } x^2 = a^2 + b^2 +$$

$$\begin{aligned} & \quad 2 \sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)} \\ &= a^2 + b^2 + 2 \sqrt{(a^2 + b^2) p - p^2} \end{aligned}$$

Linked Comprehension Type

$$1. (1) \sin \alpha = A \sin(\alpha + \beta) = A (\sin \alpha \cos \beta + \sin \beta \cos \alpha)$$

$$\Rightarrow \sin \alpha (1 - A \cos \beta) = A \sin \beta \cos \alpha \quad \dots(i)$$

$$\Rightarrow \tan \alpha = \frac{A \sin \beta}{(1 - A \cos \beta)} \quad \dots(ii)$$

$$2. (2) \tan \beta = \frac{\sin \beta}{\cos \beta} = \frac{(1 - A \cos \beta) \tan \alpha}{A \cos \beta}$$

$$= \frac{(1 - A \cos \beta) \sin \alpha}{A \cos \alpha \cos \beta} \quad [\text{from Eqs. (i) and (ii)}]$$

$$3. (3) \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{A \sin \beta}{1 - A \cos \beta} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{A \sin \beta \sin \beta}{(1 - A \cos \beta) \cos \beta}}$$

$$= \frac{A \sin \beta \cos \beta + \sin \beta - A \sin \beta \cos \beta}{\cos \beta - A \cos^2 \beta - A \sin^2 \beta}$$

$$= \frac{\sin \beta}{\cos \beta - A}$$

$$\text{Also, } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \alpha (1 - A \cos \beta)}{A \cos \alpha \cos \beta}}{1 - \frac{\sin^2 \alpha (1 - A \cos \beta)}{A \cos^2 \alpha \cos \beta}} \quad [\text{from Eq. (ii)}]$$

$$= \frac{[A \sin \alpha \cos \beta + \sin \alpha - A \sin \alpha \cos \beta] \cos \alpha}{A \cos^2 \alpha \cos \beta - \sin^2 \alpha + A \sin^2 \alpha \cos \beta}$$

$$= \frac{\sin \alpha \cos \alpha}{A \cos \beta - \sin^2 \alpha}$$

4. (4), 5. (1), 6. (2)

$$\text{We have } \tan\left(\theta + \frac{\pi}{4}\right) = 3 \tan 3\theta$$

$$\text{or } \frac{1 + \tan \theta}{1 - \tan \theta} = 3 \times \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\Rightarrow \frac{1+t}{1-t} = 3 \left(\frac{3t - t^3}{1-3t^2} \right) \quad (\text{putting } t = \tan \theta)$$

$$\text{or } 3t^4 - 6t^2 + 8t - 1 = 0$$

Hence,

$$S_1 = \text{sum of roots} = t_1 + t_2 + t_3 + t_4 = 0$$

$$S_2 = \text{sum of product of roots taken two at a time} = -2$$

$$S_3 = \text{sum of product of roots taken three at a time} = -8/3$$

$$S_4 = \text{product of all roots} = -1/3$$

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{\sum t_1 t_2 t_3}{t_1 t_2 t_3 t_4} = \frac{-8/3}{-1/3} = 8$$

7. (1), 8. (2), 9. (4).

$$\sin \alpha + \sin \beta = \frac{1}{4} \quad \dots(i)$$

$$\Rightarrow 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{4} \quad \dots(ii)$$

$$\text{where, } p = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

$$\text{or } p = \frac{1}{2} [a^2 + b^2 + (a^2 - b^2) \cos 2\alpha]$$

1. (1), (2), (4)

$$\text{Let } \tan(\theta/2) = \alpha \text{ and } \tan(\phi/2) = \beta, \text{ so that } \alpha - \beta = 2b.$$

$$\text{Also, } \cos \theta = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1 - \alpha^2}{1 + \alpha^2}$$

$$\text{and } \sin \theta = \frac{2 \tan(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{2\alpha}{1 + \alpha^2}$$

$$\text{Similarly, } \cos \phi = \frac{1 - \beta^2}{1 + \beta^2} \text{ and } \sin \phi = \frac{2\beta}{1 + \beta^2}$$

Therefore, we have from the given relations

$$(x - a) \frac{1 - \alpha^2}{1 + \alpha^2} + y \left(\frac{2\alpha}{1 + \alpha^2} \right) = a$$

$$\Rightarrow x\alpha^2 - 2y\alpha + 2a - x = 0$$

$$\text{Similarly, } x\beta^2 - 2y\beta + 2a - x = 0$$

We see that α and β are roots of the equation

$$xz^2 - 2yz + 2a - x = 0,$$

$$\text{So that } \alpha + \beta = 2y/x \text{ and } \alpha\beta = (2a - x)/x.$$

Now, from $(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$, we get

$$\Rightarrow \left(\frac{2y}{x} \right)^2 = (2b)^2 + \frac{4(2a - x)}{x}$$

$$\Rightarrow y^2 = 2ax - (1 - b^2)x^2$$

Also, from $\alpha + \beta = 2y/x$ and $\alpha - \beta = 2b$, we get

$$\alpha = y/x + b \text{ and } \beta = y/x - b$$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{1}{x} (y + bx) \text{ and } \tan \frac{\phi}{2} = \frac{1}{x} (y - bx)$$

11. (2), (3)

We have,

$$\frac{2}{\cos x} = \frac{1}{\cos(x-y)} + \frac{1}{\cos(x+y)} = \frac{2 \cos x \cos y}{\cos^2 x - \sin^2 y}$$

$$\Rightarrow \cos^2 x - \sin^2 y = \cos^2 x \cdot \cos y$$

$$\Rightarrow \cos^2 x (1 - \cos y) = \sin^2 y$$

$$\Rightarrow \cos^2 x \cdot 2 \sin^2 \frac{y}{2} = 4 \sin^2 \frac{y}{2} \cos^2 \frac{y}{2}$$

$$\Rightarrow \cos^2 x = 2 \cos^2 \frac{y}{2}$$

$$\Rightarrow \cos^2 x \sec^2 \frac{y}{2} = 2$$

$$\Rightarrow \cos x \cdot \sec \frac{y}{2} = \pm \sqrt{2}$$

12. (2), (4)

$$E = 60 \sin \alpha + p \cos \alpha$$

$$\text{Maximum value of } E = \sqrt{3600 + p^2}$$

$$\text{Minimum value of } E = -\sqrt{3600 + p^2}$$

$$\sqrt{3600 + p^2} + \sqrt{3600 + p^2} = 122$$

$$\therefore 2\sqrt{3600 + p^2} = 122$$

$$\Rightarrow p^2 = 121$$

$$\Rightarrow p = \pm 11$$

(given)

$$\Rightarrow \cos \alpha + \cos \beta = \frac{1}{3} \quad \dots(\text{iii})$$

$$\Rightarrow 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) = \frac{1}{3} \quad \dots(\text{iv})$$

Dividing Eq. (ii) by Eq. (iv), we have

$$\tan \left(\frac{\alpha + \beta}{2} \right) = \frac{3}{4}$$

$$\Rightarrow \sin(\alpha + \beta) = \frac{2 \tan \left(\frac{\alpha + \beta}{2} \right)}{1 + \tan^2 \left(\frac{\alpha + \beta}{2} \right)}$$

$$= \frac{2 \times \frac{3}{4}}{1 + \left(\frac{3}{4} \right)^2} = \frac{24}{25}$$

$$\therefore \cos(\alpha + \beta) = \frac{7}{25} \text{ and } \tan(\alpha + \beta) = \frac{24}{7}$$

10. (1), 11. (3), 12. (2).

$$\text{Let } \theta = \frac{n\pi}{7} \text{ (so that } 7\theta = n\pi, n \in \mathbb{Z})$$

$$\Rightarrow 4\theta + 3\theta = n\pi$$

$$\text{or } \tan 4\theta = \tan(n\pi - 3\theta) = -\tan 3\theta$$

$$\text{or } \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta} = -\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\text{or } \frac{4z - 4z^3}{1 - 6z^2 + z^4} = -\frac{3z - z^3}{1 - 3z^2} \quad [\text{where } \tan \theta = z \text{ (say)}]$$

$$\text{or } (4 - 4z^2)(1 - 3z^2) = -(3 - z^2)(1 - 6z^2 + z^4)$$

$$\text{or } z^6 - 21z^4 + 35z^2 - 7 = 0 \quad \dots(\text{i})$$

This is a cubic equation in z^2 , i.e., in $\tan^2 \theta$.

The roots of this equation are therefore

$$\tan^2 \pi/7, \tan^2 2\pi/7, \text{ and } \tan^2 3\pi/7. \text{ From Eq. (i),}$$

$$\text{Sum of the roots} = \frac{-(-21)}{1} = 21$$

$$\Rightarrow \tan^2 \frac{\pi}{7} + \tan^2 \frac{2\pi}{7} + \tan^2 \frac{3\pi}{7} = 21 \quad \dots(\text{ii})$$

Putting $1/y$ in place of z in Eq. (i), we get

$$-7y^6 + 35y^4 - 21y^2 + 1 = 0$$

$$\text{or } 7y^6 - 35y^4 + 21y^2 - 1 = 0 \quad \dots(\text{iii})$$

This is a cubic equation in y^2 , i.e., in $\cot^2 \theta$.

The roots of this equation are therefore $\cot^2 \pi/7, \cot^2 2\pi/7,$ and $\cot^2 3\pi/7$.

$$\text{Sum of the roots of Eq. (iii)} = \frac{35}{7} = 5$$

$$\Rightarrow \cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} = 5 \quad \dots(\text{iv})$$

By multiplying Eqs. (ii) and (iv), we get

$$\left(\tan^2 \frac{\pi}{7} + \tan^2 \frac{2\pi}{7} + \tan^2 \frac{3\pi}{7} \right) \left(\cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} \right) = 21 \times 5 = 105$$

13. (1), 14. (2), 15. (4)

$$\therefore \cos A \cos B \cos C = \frac{\sqrt{3} - 1}{8}$$

$$\sin A \sin B \sin C = \frac{3 + \sqrt{3}}{8}$$

$$\therefore \tan A \tan B \tan C = \frac{3 + \sqrt{3}}{\sqrt{3} - 1}$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C = \frac{3 + \sqrt{3}}{\sqrt{3} - 1} \quad \dots(1)$$

Now $A + B + C = \pi$

$$\therefore \cos(A + B + C) = -1$$

$$\therefore \cos A \cos B \cos C [1 - \Sigma \tan A \tan B] = -1$$

$$\therefore \frac{\sqrt{3} - 1}{8} [1 - \Sigma \tan A \tan B] = -1$$

$$\Rightarrow \Sigma \tan A \tan B = 5 + 4\sqrt{3} \quad \dots(2)$$

From (1), (2) and (3), we get

$\tan A, \tan B, \tan C$ are roots of

$$x^3 - \left(\frac{3 + \sqrt{3}}{\sqrt{3} - 1} \right) x^2 + (5 + 4\sqrt{3}) x - \frac{(3 + \sqrt{3})}{\sqrt{3} - 1} = 0$$

$$\text{or } x^3 - (2 + \sqrt{3})\sqrt{3} x^2 + (5 + 4\sqrt{3}) x - (2 + \sqrt{3})\sqrt{3} = 0$$

$$\text{or } x^3 - (3 + 2\sqrt{3}) x^2 + (5 + 4\sqrt{3}) x - (3 + 2\sqrt{3}) = 0$$

$$\text{or } (x - 1)(x - \sqrt{3})(x - (2 + \sqrt{3})) = 0$$

$$\therefore \tan A = 1, \tan B = \sqrt{3}, \tan C = 2 + \sqrt{3}$$

16. (2) 17. (3)

$$\text{We have } \sin \left(\frac{\alpha - \beta}{2} \right) + \sin \left(\frac{\alpha - \gamma}{2} \right) + \sin \left(\frac{3\alpha}{2} \right) = \frac{3}{2}$$

$$\therefore \sin \left(\frac{(\pi - (\beta + \gamma)) - \beta}{2} \right) + \sin \left(\frac{(\pi - (\gamma + \beta)) - \gamma}{2} \right)$$

$$+ \sin \left(\frac{3\pi - 3(\beta + \gamma)}{2} \right) = \frac{3}{2}$$

$$\therefore \cos \left(\frac{2\beta + \gamma}{2} \right) + \cos \left(\frac{2\gamma + \beta}{2} \right) - \cos \left(\frac{3(\beta + \gamma)}{2} \right) = \frac{3}{2}$$

$$\therefore 2 \cos \left(\frac{3}{4}(\beta + \gamma) \right) \cos \left(\frac{\beta - \gamma}{4} \right) + 1 - 2 \cos^2 \left(\frac{3}{4}(\beta + \gamma) \right) = \frac{3}{2}$$

$$\therefore 4 \cos^2 \left(\frac{3}{4}(\beta + \gamma) \right) - 4 \cos \left(\frac{3}{4}(\beta + \gamma) \right) \cos \left(\frac{\beta - \gamma}{4} \right) + 1 = 0 \quad \dots(1)$$

Above equation is quadratic in $\cos \left(\frac{3}{4}(\beta + \gamma) \right)$

Since $\cos \left(\frac{3}{4}(\beta + \gamma) \right)$ is a real number,

Discriminant $D \geq 0$

$$\therefore 16 \cos^2 \left(\frac{\beta - \gamma}{4} \right) - 16 \geq 0$$

$$\Rightarrow \cos^2 \left(\frac{\beta - \gamma}{4} \right) \geq 1$$

$$\Rightarrow \cos^2 \left(\frac{\beta - \gamma}{4} \right) = 1$$

$$\Rightarrow \beta = \gamma$$

From equation (1) for $\beta = \gamma$, we get

$$\left[2\cos\left(\frac{3}{4}(\beta + \gamma)\right) - 1 \right]^2 = 0$$

$$\Rightarrow 2\cos\frac{3}{4}(\beta + \gamma) = 1$$

$$\Rightarrow \cos\frac{3}{4}(\beta + \gamma) = \frac{1}{2}$$

$$\Rightarrow \frac{3}{4}(\beta + \gamma) = 60^\circ$$

$$\therefore \beta + \gamma = 80^\circ$$

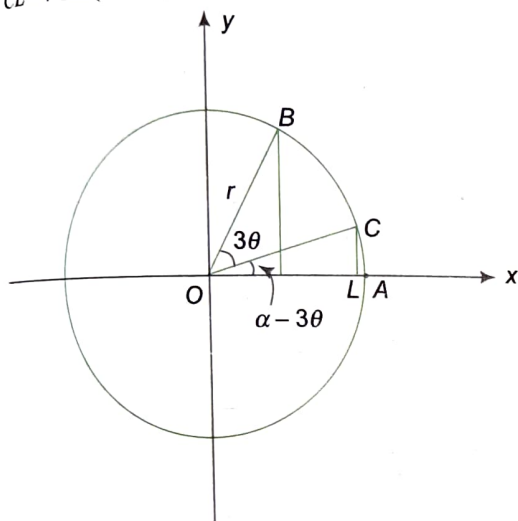
$$\therefore \alpha = 100^\circ$$

$$\therefore \beta = 40^\circ, \gamma = 40^\circ$$

19. (4), 20. (2)

$$OL = r \cos(\alpha - 3\theta)$$

$$CL = r \sin(\alpha - 3\theta)$$



$$\text{So, } \frac{\sin^3 \theta}{r \sin(\alpha - 3\theta)} = \frac{\cos^3 \theta}{r \cos(\alpha - 3\theta)} = 1$$

$$\Rightarrow \frac{\sin^3 \theta}{\sin(\alpha - 3\theta)} = \frac{\cos^3 \theta}{\cos(\alpha - 3\theta)} = r$$

Now $\cos^4 \theta - \sin^4 \theta$

$$= \cos \theta (r \cos(\alpha - 3\theta)) - \sin \theta (r \sin(\alpha - 3\theta))$$

$$\Rightarrow \cos 2\theta = r \cos(\alpha - 2\theta)$$

$$\Rightarrow \cos 2\theta = r (\cos \alpha \cos 2\theta + \sin \alpha \sin 2\theta)$$

$$\Rightarrow (1 - r \cos \alpha) \cos 2\theta = r \sin \alpha \sin 2\theta$$

$$\Rightarrow \frac{1 - r \cos \alpha}{r \sin \alpha} = \tan 2\theta \quad \dots(1)$$

$$\cos^3 \theta \sin \theta + \sin^3 \theta \cos \theta$$

$$= r [\cos(\alpha - 3\theta) \sin \theta + \sin(\alpha - 3\theta) \cos \theta]$$

$$\Rightarrow \sin \theta \cos \theta = r \sin(\alpha - 2\theta)$$

$$\Rightarrow \sin 2\theta = 2r [\sin \alpha \cos 2\theta - \cos \alpha \sin 2\theta]$$

$$\Rightarrow (1 + 2r \cos \alpha) \sin 2\theta = 2r \sin \alpha \cos 2\theta$$

$$\Rightarrow \frac{1 + 2r \cos \alpha}{2r \sin \alpha} = \cot 2\theta \quad \dots(2)$$

From (1) and (2), we get

$$\frac{1 - r \cos \alpha}{r \sin \alpha} = \frac{2r \sin \alpha}{1 + 2r \cos \alpha}$$

$$\Rightarrow 1 - r \cos \alpha + 2r \cos \alpha - 2r^2 \cos^2 \alpha = 2r^2 \sin^2 \alpha$$

$$\Rightarrow \frac{2r^2 - 1}{r} = \cos \alpha$$

Matrix Match Type

Solutions 5.57

1. $a \rightarrow q; b \rightarrow r; c \rightarrow s; d \rightarrow p$

$$\cos \theta - \sin \theta = \frac{1}{5}, \text{ where } 0 < \theta < \frac{\pi}{2} \quad \dots(i)$$

Squaring both sides of Eq. (i), we get

$$1 - \sin 2\theta = \frac{1}{25}$$

$$\text{or } \sin 2\theta = \frac{24}{25}$$

$$\text{or } \cos 2\theta = \frac{7}{25}$$

$$\begin{aligned} \text{Also, } (\cos \theta + \sin \theta)^2 &= (\cos \theta - \sin \theta)^2 + 4 \cos \theta \sin \theta \\ &= \frac{1}{25} + 2 \sin 2\theta = \frac{1}{25} + \frac{48}{25} = \frac{49}{25} \end{aligned}$$

$$\Rightarrow \cos \theta + \sin \theta = \frac{7}{5} \quad \dots(ii)$$

$$\text{or } \frac{(\cos \theta + \sin \theta)}{2} = \frac{7}{10}$$

Also solving Eqs. (i) and (ii), we get $\cos \theta = 4/5$.

2. $a \rightarrow r; b \rightarrow p; c \rightarrow q; d \rightarrow s$

$$\cos \alpha + \cos \beta = 1/2$$

$$\Rightarrow 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{2} \quad \dots(i)$$

$$\sin \alpha + \sin \beta = 1/3$$

$$\Rightarrow 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{3} \quad \dots(ii)$$

Dividing Eq. (ii) by Eq. (i), we get

$$\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{2}{3}$$

$$\Rightarrow \cos\left(\frac{\alpha + \beta}{2}\right) = \pm \frac{3}{\sqrt{13}}$$

Squaring and adding the given results, we have

$$2 + 2 \cos(\alpha - \beta) = \frac{13}{36}$$

$$\text{or } \cos(\alpha - \beta) = -\frac{59}{72}$$

$$\text{Now, } 2 \cos^2\left(\frac{\alpha - \beta}{2}\right) - 1 = \cos(\alpha - \beta)$$

$$\text{or } 2 \cos^2\left(\frac{\alpha - \beta}{2}\right) = 1 - \frac{59}{72} = \frac{13}{72}$$

$$\text{or } \cos\left(\frac{\alpha - \beta}{2}\right) = \pm \frac{\sqrt{13}}{12}$$

$$\text{or } \tan\left(\frac{\alpha - \beta}{2}\right) = \pm \sqrt{\frac{131}{13}}$$

3. $a \rightarrow s; b \rightarrow s; c \rightarrow q; d \rightarrow p$

$$a. \{ \cos(2A + \theta) + \cos(2B + \theta) \} = 2 \cos(A - B) \cos(A + B + \theta)$$

$$\begin{aligned} \text{Maximum value is } 2 \cos(A - B) \text{ when} \\ \cos(A + B + \theta) = 1 \end{aligned}$$

$$b. \{ \cos 2A + \cos 2B \} = 2 \cos(A + B) \cos(A - B)$$

$$\begin{aligned} \text{Maximum value is } 2 \cos(A - B) \text{ when} \\ \cos(A + B) = 1 \end{aligned}$$

- c. For $y = \sec x$, $x \in (0, \pi/2)$, tangent drawn to it at any point lies completely below the graph of $y = \sec x$; thus,

$$\frac{\sec 2A + \sec 2B}{2} \geq 2 \sec(A+B)$$

$$\text{or } \sec 2A + \sec 2B \geq 2 \sec(A+B)$$

Hence, the minimum value is $2 \sec(A+B)$.

d. $\sqrt{\tan \theta + \cot \theta - 2 \cos 2(A+B)}$

$$= \sqrt{(\sqrt{\tan \theta} - \sqrt{\cot \theta})^2 + 2 - 2 \cos 2(A+B)}$$

$$= \sqrt{(\sqrt{\tan \theta} - \sqrt{\cot \theta})^2 + 4 \sin^2(A+B)}$$

Minimum value occurs when $\sqrt{\tan \theta} = \sqrt{\cot \theta}$ and

$$\text{minimum value is } \sqrt{4 \sin^2(A+B)} = 2 \sin(A+B)$$

4. $a \rightarrow s$; $b \rightarrow r$; $c \rightarrow p$; $d \rightarrow q$

a. $\cos 20^\circ + \cos 80^\circ - \sqrt{3} \cos 50^\circ$
 $= 2 \cos 30^\circ \cos 50^\circ - \sqrt{3} \cos 50^\circ$
 $= \sqrt{3} \cos 50^\circ - \sqrt{3} \cos 50^\circ = 0$

b. $\cos 0^\circ + \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{7} +$
 $\cos \frac{5\pi}{7} + \cos \frac{6\pi}{7}$
 $= 1 + \left(\cos \frac{\pi}{7} + \cos \frac{6\pi}{7} \right) + \left(\cos \frac{2\pi}{7} + \cos \frac{5\pi}{7} \right)$
 $+ \left(\cos \frac{3\pi}{7} + \cos \frac{4\pi}{7} \right)$
 $= 1 + \left(\cos \frac{\pi}{7} + \cos \left(\pi - \frac{\pi}{7} \right) \right) + \left(\cos \frac{2\pi}{7} + \cos \left(\pi - \frac{2\pi}{7} \right) \right)$
 $+ \left(\cos \frac{3\pi}{7} + \cos \left(\pi - \frac{3\pi}{7} \right) \right)$
 $= 1 + 0 + 0 + 0 = 1$

c. $\cos 20^\circ + \cos 40^\circ + \cos 60^\circ - 4 \cos 10^\circ \cos 20^\circ \cos 30^\circ$
 $= 2 \cos 30^\circ \cos 10^\circ + 2 \cos^2 30^\circ - 1$
 $- 4 \cos 10^\circ \cos 20^\circ \cos 30^\circ$
 $= 2 \cos 30^\circ (\cos 10^\circ + \cos 30^\circ) - 1$
 $- 4 \cos 10^\circ \cos 20^\circ \cos 30^\circ$
 $= 2 \cos 30^\circ (2 \cos 10^\circ \cos 20^\circ) - 1$
 $- 4 \cos 10^\circ \cos 20^\circ \cos 30^\circ = -1$

d. $\cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ$
 $= \frac{1}{2} (\cos 120^\circ + \cos 80^\circ + \cos 240^\circ + \cos 40^\circ - \cos 340^\circ$
 $- \cos 60^\circ)$
 $= \frac{1}{2} \left(-\frac{1}{2} + \cos 80^\circ - \frac{1}{2} + \cos 40^\circ - \cos 340^\circ - \frac{1}{2} \right)$
 $= \frac{1}{2} \left(-\frac{3}{2} + \cos 80^\circ + \cos 40^\circ - \cos 20^\circ \right)$
 $= \frac{1}{2} \left(-\frac{3}{2} + 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ \right)$
 $= \frac{1}{2} \left(-\frac{3}{2} \right) = -\frac{3}{4}$

5. $a \rightarrow p$; $b \rightarrow p$; $c \rightarrow q$; $d \rightarrow s$

a. $x = \sin \theta$, $y = \cos \theta$

$$P = (3 \sin \theta - 4 \sin^3 \theta)^2 + (3 \cos \theta - 4 \cos^3 \theta)^2$$

$$= \sin^2 3\theta + \cos^2 3\theta = 1$$

b. On adding, we get $a = \frac{3 - \cos 4\theta + 4 \sin 2\theta}{2} = (1 + \sin 2\theta)^2$

On subtracting, we get $b = (1 - \sin 2\theta)^2$

$$\Rightarrow ab = \cos^4 2\theta \leq 1$$

c. $3 \cos \theta = x^2 - 8x + 19$

$$\Rightarrow 3 \cos \theta = (x-4)^2 + 3$$

$$\text{Now L.H.S.} = 3 \cos \theta \leq 3$$

$$\text{or L.H.S. has the greatest value } 3.$$

$$\text{But R.H.S.} = (x-4)^2 + 3 \geq 3$$

$$\text{or R.H.S. has the least value } 3.$$

$$\text{Hence, L.H.S.} = \text{R.H.S.}$$

$$\text{when } 3 \cos \theta = (x-4)^2 + 3 = 3$$

$$\Rightarrow \cos \theta = 1 \text{ and } x-4 = 0$$

$$\Rightarrow \theta = 2n\pi \text{ and } x = 4, \text{ where } n \in \mathbb{Z}.$$

d. Let $\lambda = \tan \theta$

$$\therefore x = 2 \sin 2\theta \text{ and } y = 2 \cos 2\theta$$

$$E = x^2 - xy + y^2 = 4 - 4 \sin 2\theta \cos 2\theta = 4 - 2 \sin 4\theta$$

$$E \in [2, 6] \Rightarrow a + b = 8$$

6. $a \rightarrow q$; $b \rightarrow p$; $c \rightarrow s$; $d \rightarrow r$

a. $9 + 16 + 24 \sin(A+B) = 37$ (on squaring and adding)

$$24 \sin(A+B) = 12$$

$$\sin(A+B) = \frac{1}{2} \Rightarrow \sin C = \frac{1}{2}$$

$$C = 30^\circ \text{ or } 150^\circ$$

$$\Rightarrow C = 30^\circ$$

b. $(\sin A + \sin B)^2 - \sin^2 C = 3 \sin A \sin B$

$$\text{or } \sin^2 A - \sin^2 C + \sin^2 B = \sin A \sin B$$

$$\text{or } \sin(A+C) \sin(A-C) + \sin^2 B = \sin A \sin B$$

$$\text{or } \sin B [\sin(A-C) + \sin(A+C)] = \sin A \sin B$$

$$\text{or } 2 \sin A \cos C = \sin A (\text{as } \sin B \neq 0)$$

$$\text{or } \cos C = 1/2$$

$$\text{or } C = 60^\circ$$

c. $2 \sin x \cos x [4 \cos^4 x - 4 \sin^4 x] = 1$

$$\text{or } (\sin 2x) [2(\cos^2 x + \sin^2 x)][2 \cos^2 x - 2 \sin^2 x] = 1$$

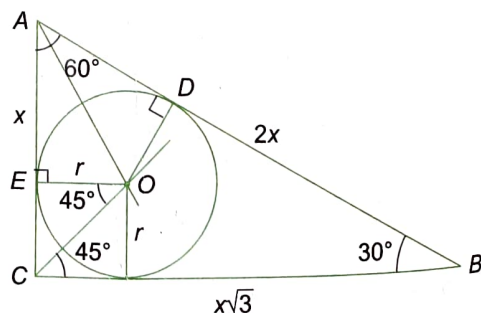
$$\text{or } (\sin 2x) 2 \times 2 \cos 2x = 1$$

$$\text{or } 2 \sin 4x = 1$$

$$\text{or } \sin 4x = \frac{1}{2} \text{ or } 4x = 30^\circ \text{ or } x = 7.5^\circ$$

d. Obviously, $AEOD$ is a cyclic quadrilateral, we have

$$\angle COD = 120^\circ + 45^\circ = 165^\circ$$



7. $a \rightarrow r$; $b \rightarrow q$; $c \rightarrow s$; $d \rightarrow p$

a. We have $\frac{4 + \sec 20^\circ}{\operatorname{cosec} 20^\circ} = \frac{\sin 20^\circ}{\cos 20^\circ} (4 \cos 20^\circ + 1)$

$$= \frac{2 \sin 40^\circ + \sin 20^\circ}{\cos 20^\circ}$$

$$\begin{aligned}
 &= \frac{\sin 40^\circ + (\sin 40^\circ + \sin 20^\circ)}{\cos 20^\circ} \\
 &= \frac{\sin 40^\circ + 2\sin 30^\circ \cos 10^\circ}{\cos 20^\circ} \\
 &= \frac{\sin 40^\circ + \sin 80^\circ}{\cos 20^\circ} \\
 &= \frac{2\sin 60^\circ \cos 20^\circ}{\cos 20^\circ} \\
 &= 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. Let } f(x) &= \frac{2\cos^2 x + 8\sin^2 x}{2\sin 2x}, f(x) > 0 \\
 &= \frac{\cot x + 4\tan x}{2} \\
 &= \frac{(\sqrt{\cot x} - 2\sqrt{\tan x})^2}{2} + 2
 \end{aligned}$$

$$\therefore f(x)|_{\min} = 2$$

$$\begin{aligned}
 \text{c. } \frac{8\sin 40^\circ \cdot \sin 50^\circ \cdot \tan 10^\circ}{\cos 80^\circ} &= \frac{4(2\sin 50^\circ \cdot \sin 40^\circ) \sin 10^\circ}{\sin 10^\circ \cdot \cos 10^\circ} \\
 &= \frac{4(\cos 10^\circ - \cos 90^\circ) \sin 10^\circ}{\sin 10^\circ \cdot \cos 10^\circ} = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{d. L.H.S} &= \frac{2\sin 6A}{\sin 2A} = \frac{2\sin 3(2A)}{\sin 2A} \\
 &= \frac{2[3\sin 2A - 4\sin^3 2A]}{\sin 2A} \\
 &= 2(3 - 4\sin^2 2A) \\
 &= 2(3 - 2(1 - \cos 4A)) \\
 &= 2 + 4\cos 4A
 \end{aligned}$$

$$\therefore a = 2, b = 4$$

$$\therefore a^2/b = 1$$

$$\text{a. } 37^\circ = 45^\circ - 8^\circ$$

$$\Rightarrow \tan 37^\circ = \frac{1 - \tan 8^\circ}{1 + \tan 8^\circ}$$

$$\Rightarrow 1 + \tan 37^\circ = \frac{2}{1 + \tan 8^\circ}$$

$$\text{Similarly, } 1 + \tan 23^\circ = \frac{2}{1 + \tan 22^\circ}$$

$$\Rightarrow \frac{(1 + \tan 8^\circ)(1 + \tan 37^\circ)}{(1 + \tan 22^\circ)(1 + \tan 23^\circ)} = 1$$

$$\begin{aligned}
 \text{b. } 2 \sum \left(\frac{\cot A + \cot B}{\tan A + \tan B} \right) &= 2 \sum \frac{\cos A \cos B}{\sin A \sin B} \\
 &= 2 \sum \frac{\cos A \cos B \sin C}{\sin A \sin B \sin C} \\
 &= 2(\cot A \cot B \cot C \sum \tan C) \\
 &= 2\cot A \cot B \cot C (\tan A \tan B \tan C) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } |\cos(\cos 3x + 3\cos x)| &= |\cos(4\cos^3 x)| \geq 0 \\
 \therefore \text{minimum value is } 0
 \end{aligned}$$

$$\text{d. } 16 \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5}$$

$$\begin{aligned}
 &= 16 \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \left(\pi - \frac{2\pi}{5} \right) \sin \left(\pi - \frac{\pi}{5} \right) \\
 &= 16 \sin^2 \frac{\pi}{5} \cdot \sin^2 \frac{2\pi}{5} \\
 &= 4 \left[\cos \frac{\pi}{5} - \cos \frac{3\pi}{5} \right]^2 \\
 &= 4 \left[\cos \frac{\pi}{5} + \sin \frac{\pi}{10} \right]^2 \\
 &= 4 \left[\frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4} \right]^2 = 5
 \end{aligned}$$

9. (3)

$$\begin{aligned}
 \text{a. } \cos 2A + \cos 2B + \cos 2C &= -1 \\
 \Rightarrow -1 - 4\cos A \cos B \cos C &= -1 \\
 \Rightarrow \cos A \cos B \cos C &= 0 \\
 \Rightarrow \cos A = 0 \text{ or } \cos B = 0 \text{ or } \cos C = 0
 \end{aligned}$$

Thus, triangle is right angled.

$$\text{b. } \tan A > 0, \tan B > 0 \text{ and } \tan A \tan B < 1$$

$$\tan C = \frac{\tan A + \tan B}{\tan A \tan B - 1} < 0$$

$$\therefore \angle C > 90^\circ$$

$$\begin{aligned}
 \text{c. } \cos^3 A + \cos^3 B + \cos^3 C &= 3 \cos A \cos B \cos C \\
 \Rightarrow (\cos A + \cos B + \cos C)(\cos^2 A + \cos^2 B + \cos^2 C - \cos A \cos B - \cos B \cos C - \cos A \cos C) &= 0 \\
 \Rightarrow \cos A + \cos B + \cos C &= 0 \\
 \text{or } \cos^2 A + \cos^2 B + \cos^2 C - \cos A \cos B - \cos B \cos C - \cos A \cos C &= 0
 \end{aligned}$$

$$\Rightarrow 1 + 4 \sin(A/2) \sin(B/2) \sin(C/2) = 0$$

$$\text{or } \cos A = \cos B = \cos C$$

$$\Rightarrow \sin(A/2) \sin(B/2) \sin(C/2) = -1/4 \text{ (which is not possible as } \sin(A/2), \sin(B/2), \sin(C/2) > 0)$$

$$\text{or } \cos A = \cos B = \cos C$$

Thus, triangle is equilateral.

$$\text{d. } \cot A > 0, \cot B > 0 \text{ and } \cot A \cot B < 1$$

$$\therefore \tan A > 0, \tan B > 0 \text{ and } \tan A \tan B > 1$$

$$\therefore \tan C > 0$$

$$\therefore \angle C < 90^\circ$$

Numerical Value Type

$$\begin{aligned}
 \text{1. (0.5) } f(\theta) &= \frac{1 - \sin 2\theta + \cos 2\theta}{2 \cos 2\theta} \\
 &= \frac{(\cos \theta - \sin \theta)^2 + (\cos^2 \theta - \sin^2 \theta)}{2(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)} \\
 &= \frac{\cos \theta}{\cos \theta + \sin \theta} \\
 &= \frac{1}{1 + \tan \theta}
 \end{aligned}$$

$$\begin{aligned}
 f(11^\circ) \cdot f(34^\circ) &= \frac{1}{(1 + \tan 11^\circ)} \times \frac{1}{(1 + \tan 34^\circ)} \\
 &= \frac{1}{(1 + \tan 11^\circ)} \times \frac{1}{(1 + \tan(45^\circ - 11^\circ))}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(1 + \tan 11^\circ)} \times \frac{1}{\left(1 + \frac{1 - \tan 11^\circ}{1 + \tan 11^\circ}\right)} \\
 &= \frac{1}{(1 + \tan 11^\circ)} \times \frac{(1 + \tan 11^\circ)}{2} \\
 &= \frac{1}{2}
 \end{aligned}$$

2. (625) $f(x) = 2(7 \cos x + 24 \sin x)(7 \sin x - 24 \cos x)$

Let $r \cos \theta = 7$; $r \sin \theta = 24$

$\therefore r^2 = 625$; $\tan \theta = \frac{24}{7}$

$\therefore f(x) = 2r \cos(x - \theta) \times r \sin(x - \theta)$
 $= r^2(\sin 2(x - \theta))$

$f(x)_{\max} = 25^2$

3. (1) $\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) = -\frac{b}{a}$

$\tan\left(\frac{A}{2}\right) \times \tan\left(\frac{B}{2}\right) = \frac{c}{a}$

$A + B = 90^\circ$ or $\frac{A+B}{2} = 45^\circ$

$\Rightarrow \tan\left(\frac{A+B}{2}\right) = 1 = \frac{-\frac{b}{a}}{1 - \frac{c}{a}}$

or $1 - \frac{c}{a} = -\frac{b}{a}$

or $a + b = c$

or $\frac{a+b}{c} = 1$

4. (1) Let $x + 5 = 14 \cos \theta$ and $y - 12 = 14 \sin \theta$

$\therefore x^2 + y^2 = (14 \cos \theta - 5)^2 + (14 \sin \theta + 12)^2$
 $= 196 + 25 + 144 + 28(12 \sin \theta - 5 \cos \theta)$
 $= 365 + 28(12 \sin \theta - 5 \cos \theta)$

$\therefore \sqrt{x^2 + y^2} \Big|_{\min} = \sqrt{365 - 28 \times 13}$
 $= \sqrt{365 - 364} = 1$

5. (294) $\cot x + \cot y = 49$

or $\frac{1}{\tan x} + \frac{1}{\tan y} = 49$

or $\frac{\tan y + \tan x}{\tan x \tan y} = 49$

or $\tan x \tan y = \frac{\tan x + \tan y}{49} = \frac{42}{49} = \frac{6}{7}$

$\therefore \tan(x + y) = \frac{42}{1 - (6/7)} = \frac{42}{1/7} = 294$

6. (7) From the given equations, we have

$(2 \cos a + 9 \cos d)^2 = (6 \cos b + 7 \cos c)^2$

and $(2 \sin a - 9 \sin d)^2 = (6 \sin b - 7 \sin c)^2$

Adding, we have $36 \cos(a + d) = 84 \cos(b + c)$

or $\frac{\cos(a + d)}{\cos(b + c)} = \frac{7}{3}$

7. (8) Since $\cos A + \cos B = 0$

$\Rightarrow A + B = \pi$

$\therefore B = \pi - A$

$\Rightarrow \sin A + \sin(\pi - A) = 1$

$\Rightarrow \sin A = \frac{1}{2}$

$\Rightarrow A = 30^\circ$ and $B = 150^\circ$ or $A = 150^\circ$ and $B = 30^\circ$

$\Rightarrow 12 \cos 60^\circ + 4 \cos 300^\circ = 8$

8. (4) $5 \times \frac{2 \tan \beta}{1 + \tan^2 \beta} = 3 \times \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$

or $\frac{5 \tan \beta}{1 + \tan^2 \beta} = \frac{3 \tan \alpha}{1 + \tan^2 \alpha}$ (i)

Substituting $\tan \beta = 3 \tan \alpha$, we have

$\frac{5 \times 3 \tan \alpha}{1 + 9 \tan^2 \alpha} = \frac{3 \tan \alpha}{1 + \tan^2 \alpha}$

or $5 + 5 \tan^2 \alpha = 1 + 9 \tan^2 \alpha$

or $4 \tan^2 \alpha = 4$

or $\tan \alpha = 1$, i.e., $\tan \beta = 3$

$\therefore \tan \alpha + \tan \beta = 4$

9. (4) Let $\theta = \frac{\pi}{16}$ or $8\theta = \frac{\pi}{2}$

$y = \tan \theta + \tan 5\theta + \tan 9\theta + \tan 13\theta$

$\therefore y = (\tan \theta - \cot \theta) + (\tan 5\theta - \cot 5\theta)$

[As $\tan 13\theta = \tan(8\theta + 5\theta) = -\cot 5\theta$ and
 $\tan 9\theta = \tan(8\theta + \theta) = -\cot \theta$]

$= (\tan \theta - \cot \theta) + (\cot 3\theta - \tan 3\theta)$

$= \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} + \frac{\cos^2 3\theta - \sin^2 3\theta}{\sin 3\theta \cos 3\theta}$

$\Rightarrow y = 2 \left[\frac{\cos 6\theta}{\sin 6\theta} - \frac{\cos 2\theta}{\sin 2\theta} \right]$

$= 2 \left[\frac{\sin 2\theta \cos 6\theta - \cos 2\theta \sin 6\theta}{\sin 6\theta \sin 2\theta} \right]$

$= -2 \left[\frac{\sin 4\theta}{\cos 2\theta \sin 2\theta} \right] = -4 \quad \left(\because 6\theta = \frac{\pi}{2} - 2\theta \right)$

Hence, absolute value is 4.

10. (2) $\cos 290^\circ = \sin 20^\circ$; $\sin 250^\circ = -\sin 70^\circ = -\cos 20^\circ$

$\Rightarrow \frac{1}{\sin 20^\circ} - \frac{1}{\sqrt{3} \cos 20^\circ}$
 $= \frac{\sqrt{3} \cos 20^\circ - \sin 20^\circ}{\sqrt{3} \sin 20^\circ \cos 20^\circ}$
 $= \frac{2[\sin 60^\circ \cos 20^\circ - \sin 20^\circ \cos 60^\circ]}{\sqrt{3} \sin 20^\circ \cos 20^\circ}$
 $= \frac{4 \sin 40^\circ}{\sqrt{3} \sin 40^\circ} = \frac{4\sqrt{3}}{3}$

Hence, the greatest integer less than or equal to is 2.

11. (4) $\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)$
 $= 1 - 3 \sin^2 x \cos^2 x = 1 - \frac{3(\sin 2x)^2}{4}$

$\Rightarrow y = \frac{4}{4 - 3(\sin 2x)^2}$

$\Rightarrow y_{\max} = \frac{4}{4 - 3(1)} = 4$

$$\begin{aligned}
 &= \left[\frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}} \right]^2 + (\sin x - \cos x)^2 \\
 &= \frac{3}{2}(1 - \sin 2x)
 \end{aligned}$$

Hence, Maximum value = $\frac{3}{2}(1 - (-1)) = 3$

$$\begin{aligned}
 &\frac{1}{\sin 10^\circ} + \frac{1}{\sin 50^\circ} - \frac{1}{\sin 70^\circ} \\
 &= \frac{1}{\cos 80^\circ} + \frac{1}{\cos 40^\circ} - \frac{1}{\cos 20^\circ} \\
 &= \frac{\cos 40^\circ \cos 20^\circ + \cos 80^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ} \\
 &= 8[\cos 20^\circ(\cos 40^\circ + \cos 80^\circ) - \cos 40^\circ \cos 80^\circ] \\
 &= 8[2\cos 20^\circ \cos 60^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ] \\
 &= 4[2\cos^2 20^\circ - 2\cos 40^\circ \cos 80^\circ] \\
 &= 4[1 + \cos 40^\circ - (\cos 120^\circ + \cos 40^\circ)] \\
 &= 4 \times \frac{3}{2} = 6
 \end{aligned}$$

$$\begin{aligned}
 &\text{In } \Delta ABC, \tan A + \tan B + \tan C = \tan A \tan B \tan C \\
 &\Rightarrow x + x + 1 + 1 - x = x(1 + x)(1 - x) \\
 &\text{or } 2 + x = x - x^3 \\
 &\text{or } x^3 = -2 \Rightarrow x = -2^{1/3} \\
 &\Rightarrow \tan A = x < 0 \Rightarrow A \text{ is obtuse} \\
 &\Rightarrow \tan B = x + 1 = 1 - 2^{1/3} < 0
 \end{aligned}$$

Hence, A and B are obtuse, which is not possible in a triangle.

Hence, no such triangle can exist.

$$\text{15. (12) Given } \log_{10} \left(\frac{\sin 2x}{2} \right) = -1$$

$$\text{or } \frac{\sin 2x}{2} = \frac{1}{10}$$

$$\text{or } \sin 2x = \frac{1}{5}$$

$$\text{Also } \log_{10}(\sin x + \cos x) = \frac{\log_{10} \left(\frac{n}{10} \right)}{2}$$

$$\text{or } \log_{10}(\sin x + \cos x)^2 = \log_{10} \left(\frac{n}{10} \right)$$

$$\text{or } 1 + \sin 2x = \frac{n}{10}$$

$$\text{or } 1 + \frac{1}{5} = \frac{n}{10}$$

$$\text{or } \frac{6}{5} = \frac{n}{10}$$

$$\therefore n = 12$$

$$\begin{aligned}
 &\frac{2\sin 4^\circ \cos 3^\circ + 2\sin 4^\circ \cos 1^\circ}{\cos 1^\circ \cos 2^\circ \sin 4^\circ} \\
 &= \frac{2\sin 4^\circ [\cos 3^\circ + \cos 1^\circ]}{\cos 1^\circ \cos 2^\circ \sin 4^\circ} \\
 &= \frac{4\cos 2^\circ \cos 1^\circ}{\cos 1^\circ \cos 2^\circ} = 4
 \end{aligned}$$

$$17. (7) \quad 2\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right) \sin \frac{C}{2} = \frac{1}{16}$$

$$\text{or } \sin \frac{C}{2} \left(\frac{1}{2} - \sin \frac{C}{2} \right) = \frac{1}{16}$$

$$\text{or } \sin^2 \frac{C}{2} - \frac{1}{2} \sin \frac{C}{2} + \frac{1}{16} = 0$$

$$\text{or } \left(\frac{1}{4} - \sin \frac{C}{2} \right)^2 = 0$$

$$\text{or } \sin \frac{C}{2} = \frac{1}{4}$$

$$\text{or } \cos C = 1 - 2\sin^2 \frac{C}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

$$18. (3) \quad \tan x = 2t, \tan y = 3t, \tan z = 5t$$

$$\text{Also } x + y + z = \pi$$

$$\therefore \tan x + \tan y + \tan z = \tan x \tan y \tan z$$

$$\Rightarrow t^2 = \frac{1}{3}$$

$$\Rightarrow \tan^2 x + \tan^2 y + \tan^2 z = t^2(4 + 9 + 25) = 38t^2,$$

$$\therefore K = 3$$

$$19. (0.5) \quad 4\sin^3 x \cos 3x + 4\cos^3 x \sin 3x = \frac{3}{2}$$

$$\text{or } (3\sin x - \sin 3x) \cos 3x + (3\cos x + \cos 3x) \sin 3x = \frac{3}{2}$$

$$\text{or } 3[\sin x \cos 3x + \cos x \sin 3x] = \frac{3}{2}$$

$$\text{or } \sin 4x = \frac{1}{2}$$

$$\begin{aligned}
 20. (2) \quad \operatorname{cosec} 10^\circ - 4\sin 70^\circ &= \frac{1 - 4\sin 70^\circ \sin 10^\circ}{\sin 10^\circ} \\
 &= \frac{1 - 2(\cos 60^\circ - \cos 80^\circ)}{\sin 10^\circ} \\
 &= \frac{1 - 2\left(\frac{1}{2} - \sin 10^\circ\right)}{\sin 10^\circ} \\
 &= \frac{1 - 1 + 2\sin 10^\circ}{\sin 10^\circ} = 2
 \end{aligned}$$

$$21. (1) \quad \tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$$

$$\Rightarrow x + 2x + 3x = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow x = n\pi/6, n \in \mathbb{Z}$$

$$|\sin 3x + \cos 3x| = 1$$

$$\begin{aligned}
 22. (2) \quad 16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta - \cos \frac{5\pi}{8} \right) \\
 \left(\cos \theta - \cos \frac{7\pi}{8} \right) \\
 = 16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta - \cos \frac{7\pi}{8} \right) \times \\
 \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta - \cos \frac{5\pi}{8} \right) \\
 = 16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta + \cos \frac{\pi}{8} \right) \times \\
 \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta + \cos \frac{3\pi}{8} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= 16 \left(\cos^2 \theta - \cos^2 \frac{\pi}{8} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{8} \right) \\
 &= 16 \left(\cos^2 \theta - \cos^2 \frac{\pi}{8} \right) \left(\cos^2 \theta - \sin^2 \frac{\pi}{8} \right) \\
 &= 16 \left(\cos^4 \theta - \cos^2 \theta + \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \right) \\
 &= 16 \left(\cos^4 \theta - \cos^2 \theta + \frac{1}{8} \right) \\
 &= 16 \left(-\cos^2 \theta \sin^2 \theta + \frac{1}{8} \right) \\
 &= 16 \left(\frac{-\sin^2 2\theta}{4} + \frac{1}{8} \right) \\
 &= 16 \left(\frac{1 - 2 \sin^2 2\theta}{8} \right) \\
 &= \frac{16 \cos 4\theta}{8}
 \end{aligned}$$

23. (1) $\ln 6 = \ln 3 + \ln 2$

If $\gamma = \alpha + \beta$, then

$$\tan \gamma - \tan \alpha - \tan \beta = \tan \gamma \cdot \tan \alpha \cdot \tan \beta$$

$$\Rightarrow \frac{\tan(\ln 6) \cdot \tan(\ln 2) \cdot \tan(\ln 3)}{\tan(\ln 6) - \tan(\ln 2) - \tan(\ln 3)} = 1$$

24. (2) Given $\cot(\theta - \alpha)$, $3 \cot \theta$, $\cot(\theta + \alpha)$ are in A.P.

$$\Rightarrow 6 \cot \theta = \cot(\theta - \alpha) + \cot(\theta + \alpha)$$

$$\Rightarrow \frac{6 \cos \theta}{\sin \theta} = \frac{\sin 2\theta}{\sin(\theta + \alpha) \sin(\theta - \alpha)}$$

$$\Rightarrow 6 \cos \theta \{\sin^2 \theta - \sin^2 \alpha\} = 2 \sin^2 \theta \cos \theta$$

$$\Rightarrow 3(\sin^2 \theta - \sin^2 \alpha) = \sin^2 \theta$$

$$\Rightarrow 2 \sin^2 \theta = 3 \sin^2 \alpha$$

$$\therefore \frac{2 \sin^2 \theta}{3 \sin^2 \alpha} = 1$$

25. (1) $\frac{2 \sin x}{\sin 3x} + \frac{\tan x}{\tan 3x} = \frac{2 \sin x}{\sin x \cdot (3 - 4 \sin^2 x)} + \frac{\tan x}{(3 \tan x - \tan^3 x)}$

$$= \frac{2}{3 - 4 \sin^2 x} + \frac{(1 - 3 \tan^2 x)}{(3 - \tan^2 x)}$$

$$= \frac{2}{3 - 4 \sin^2 x} + \frac{1 - \left(\frac{3 \sin^2 x}{1 - \sin^2 x} \right)}{3 - \left(\frac{\sin^2 x}{1 - \sin^2 x} \right)}$$

$$= \frac{2}{3 - 4 \sin^2 x} + \frac{(1 - 4 \sin^2 x)}{(3 - 4 \sin^2 x)}$$

$$= \frac{(3 - 4 \sin^2 x)}{(3 - 4 \sin^2 x)} = 1$$

26. (3) $(2 - \cos 2A)(2 - \cos 2B)$

$$= (1 + 2 \sin^2 A)(1 + 2 \sin^2 B)$$

$$= 1 + 2(\sin^2 A + \sin^2 B) + 4 \sin^2 A \sin^2 B$$

...(1)

$$\text{Now, } \cot^2 A \cot^2 B = 3$$

$$\Rightarrow \cos^2 A \cos^2 B = 3 \sin^2 A \sin^2 B$$

$$\Rightarrow (1 - \sin^2 A)(1 - \sin^2 B) = 3 \sin^2 A \sin^2 B$$

$$\Rightarrow \sin^2 A + \sin^2 B + 2 \sin^2 A \sin^2 B = 1$$

From (1), we get

$$(2 - \cos 2A)(2 - \cos 2B) = 1 + 2 = 3$$

27. (6) $x = \cot \frac{11\pi}{8} = \cot \left(\pi + \frac{3\pi}{8} \right) = \cot \frac{3\pi}{8} = \sqrt{2} - 1$

$$\Rightarrow (x + 1)^2 = 2$$

$$\therefore x^2 + 2x - 1 = 0$$

$$\text{Now, } f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$$

$$= x^2(x^2 + 2x - 1) + 2x^3 + 3x^2 - 4x + 7$$

$$= 0 + 2x^3 + 3x^2 - 4x + 7$$

$$= 2x(x^2 + 2x - 1) - x^2 - 2x + 7$$

$$= -x^2 - 2x + 7$$

$$= -(x^2 + 2x - 1) + 6 = 6$$

28. (1) $\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ - \sin^2 9^\circ - \sin^2 18^\circ$
 $= \sin^2 12^\circ + \sin^2 21^\circ + (\sin^2 39^\circ - \sin^2 9^\circ)$
 $\quad \quad \quad + (\sin^2 48^\circ - \sin^2 18^\circ)$
 $= 1 - (\cos^2 12^\circ - \sin^2 21^\circ) + \sin 48^\circ \sin 30^\circ$
 $\quad \quad \quad + \sin 66^\circ \sin 30^\circ$

$$= 1 - \cos 33^\circ \cos 9^\circ + \frac{1}{2} \times 2 \sin 57^\circ \cos 9^\circ$$

$$= 1 - \cos 33^\circ \cos 9^\circ + \cos 33^\circ \cos 9^\circ = 1$$

29. (1) $f(n\theta) = \frac{2 \sin 2\theta}{\cos 2\theta - \cos 4n\theta}$

$$= \frac{2 \sin 2\theta}{2 \sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin((2n+1)\theta - (2n-1)\theta)}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin(2n+1)\theta \cos(2n-1)\theta - \cos(2n+1)\theta \sin(2n-1)\theta}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \cot(2n-1)\theta - \cot(2n+1)\theta$$

$$\therefore f(\theta) + f(2\theta) + f(3\theta) + \dots + f(n\theta)$$

$$= \cot \theta - \cot(2n+1)\theta$$

$$= \frac{\sin(2n+1)\theta \cos \theta - \cos(2n+1)\theta \sin \theta}{\sin(2n+1)\theta \sin \theta}$$

$$= \frac{\sin 2n\theta}{\sin(2n+1)\theta \sin \theta}$$

30. (6) According to the given question, we have to express L.H.S. in the form

$$C_0 + C_1 \cos x + C_2 \cos 2x + \dots + C_n \cos nx$$

Now,

$$\sin^3 x \sin 3x = \frac{3 \sin x - \sin 3x}{4} \sin 3x$$

$$= \frac{3 \sin x \sin 3x - \sin^2 3x}{4}$$

$$= \frac{3(\cos 2x - \cos 4x) - (1 - \cos 6x)}{8}$$

Hence, $n = 6$.

$$\sec \alpha = \frac{\sec(\alpha - 2\beta) + \sec(\alpha + 2\beta)}{2}$$

$$\Rightarrow \frac{2}{\cos \alpha} = \frac{\cos(\alpha + 2\beta) + \cos(\alpha - 2\beta)}{\cos(\alpha - 2\beta)\cos(\alpha + 2\beta)}$$

$$\Rightarrow \cos 2\alpha + \cos 4\beta = \cos \alpha (2 \cos \alpha \cdot \cos 2\beta)$$

$$\Rightarrow 2 \cos^2 \alpha - 1 + 2 \cos^2 2\beta - 1 = 2 \cos^2 \alpha \cos 2\beta$$

$$\Rightarrow \cos^2 \alpha (1 - \cos 2\beta) + (\cos 2\beta + 1) (\cos 2\beta - 1) = 0$$

$$\Rightarrow (1 - \cos 2\beta)(\cos^2 \alpha - \cos 2\beta - 1) = 0$$

$$\Rightarrow \cos^2 \alpha = \cos 2\beta + 1$$

$$\Rightarrow \cos^2 \alpha = 2 \cos^2 \beta$$

$$\Rightarrow 1 - \sin^2 \alpha = 2(1 - \sin^2 \beta)$$

$$\Rightarrow 2 \sin^2 \beta - \sin^2 \alpha = 1$$

$$\tan \theta = K$$

$$\Rightarrow \frac{3 \tan \frac{\theta}{3} - \tan^3 \frac{\theta}{3}}{1 - 3 \tan^2 \frac{\theta}{3}} = K$$

$$\Rightarrow \tan^3 \frac{\theta}{3} - 3K \tan^2 \frac{\theta}{3} - 3 \tan \frac{\theta}{3} + K = 0$$

$$\text{Roots of this equation are } \tan \frac{A}{3}, \tan \frac{B}{3}, \tan \frac{C}{3}.$$

$$\therefore \text{Sum of roots taken two at a time} = \sum \tan \frac{A}{3} \tan \frac{B}{3} = -3.$$

$$\left(\sin \frac{\pi}{9} \right) \left(4 + \sec \frac{\pi}{9} \right)$$

$$= \sin 20^\circ \left(4 + \frac{1}{\cos 20^\circ} \right)$$

$$= \frac{4 \sin 20^\circ \cos 20^\circ + \sin 20^\circ}{\cos 20^\circ}$$

$$= \frac{2 \sin 40^\circ + \sin 20^\circ}{\cos 20^\circ}$$

$$= \frac{\sin 40^\circ + (\sin 40^\circ + \sin 20^\circ)}{\cos 20^\circ}$$

$$= \frac{\sin 40^\circ + 2 \sin 30^\circ \cos 10^\circ}{\cos 20^\circ}$$

$$= \frac{\sin 40^\circ + \cos 10^\circ}{\cos 20^\circ}$$

$$= \frac{\cos 50^\circ + \cos 10^\circ}{\cos 20^\circ}$$

$$= \frac{2 \cos 30^\circ \cos 20^\circ}{\cos 20^\circ}$$

$$= \sqrt{3}$$

$$(2) \text{ Given expression is } \left(\frac{\sin 33^\circ}{\sin 11^\circ \sin(60^\circ - 11^\circ) \sin(60^\circ + 11^\circ)} \right)^2 +$$

$$\left(\frac{\cos 33^\circ}{\cos 11^\circ \cos(60^\circ - 11^\circ) \cos(60^\circ + 11^\circ)} \right)^2$$

$$= \left(\frac{\sin 33^\circ}{\frac{1}{4} \sin 33^\circ} \right)^2 + \left(\frac{\cos 33^\circ}{\frac{1}{4} \cos 33^\circ} \right)^2$$

$$= 16 + 16 = 32$$

$$35. (0) \sin \theta + \sin \left(\theta + \frac{2\pi}{3} \right) + \sin \left(\theta + \frac{4\pi}{3} \right) = 0$$

$$\Rightarrow \sin^3 \theta + \sin^3 \left(\theta + \frac{2\pi}{3} \right) + \sin^3 \left(\theta + \frac{4\pi}{3} \right) \\ = 3 \left(\sin \theta \cdot \sin \left(\theta + \frac{2\pi}{3} \right) \cdot \sin \left(\theta + \frac{4\pi}{3} \right) \right)$$

$$f(\theta) = \frac{-3}{4} \sin 3\theta.$$

$$\therefore f\left(\frac{\pi}{18}\right) + f\left(\frac{7\pi}{18}\right) = \frac{-3}{4} \left[\sin \frac{\pi}{6} + \sin \frac{7\pi}{6} \right] = 0$$

$$36. (0.5) \text{ Let } \frac{A}{2} = 22^\circ, \frac{B}{2} = 33^\circ, \frac{C}{2} = 35^\circ$$

$$\Rightarrow A + B + C = 180^\circ$$

$$\text{In } \triangle ABC, \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}$$

$$= 2 + 2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}$$

$$\text{Hence, } \frac{1 + \sin 22^\circ \sin 33^\circ \sin 35^\circ}{\cos^2 22^\circ + \cos^2 33^\circ + \cos^2 35^\circ} = \frac{1}{2}$$

$$37. (3) A + B = \pi/3$$

$$\Rightarrow \tan(A + B) = \sqrt{3}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \sqrt{3}$$

$$\Rightarrow \frac{\tan A + y}{1 - y} = \sqrt{3}$$

(Where $y = \tan A \tan B$)

$$\Rightarrow \tan^2 A + \sqrt{3}(y - 1) \tan A + y = 0$$

For real values of $\tan A$,

$$3(y - 1)^2 - 4y = 0$$

$$\Rightarrow 3y^2 - 10y + 3 \geq 0$$

$$\Rightarrow (y - 3)(3y - 1) \geq 0$$

$$\Rightarrow y \leq \frac{1}{3} \text{ or } y \geq 3$$

But $A, B > 0$ and $A + B = \pi/3$

$$\Rightarrow A, B < \pi/3$$

$$\Rightarrow \tan A \tan B < 3 \text{ so, } y \geq 3 \text{ is not a possibility.}$$

Therefore, $y \leq \frac{1}{3}$ i.e., max. value of y is $1/3$.

$$38. (2) \text{ We have}$$

$$\frac{\sin^3 \theta}{\sin(2\theta + \alpha)} = \frac{\cos^3 \theta}{\cos(2\theta + \alpha)} = k \text{ (let)}$$

$$\Rightarrow \frac{\sin^4 \theta}{\sin \theta \sin(2\theta + \alpha)} = \frac{\cos^4 \theta}{\cos \theta \cos(2\theta + \alpha)} = k$$

$$\Rightarrow \cos^4 \theta - \sin^4 \theta = k[\cos \theta \cos(2\theta + \alpha) - \sin \theta \sin(2\theta + \alpha)]$$

$$\Rightarrow \cos 2\theta = k \cos(3\theta + \alpha) \quad \dots(1)$$

Also,

$$\frac{\sin^3 \theta \cos \theta}{\sin(2\theta + \alpha) \cos \theta} = \frac{\sin \theta \cos^3 \theta}{\sin \theta \cos(2\theta + \alpha)} = k$$

$$\Rightarrow \sin^3 \theta \cos \theta + \sin \theta \cos^3 \theta \\ = k(\sin(2\theta + \alpha) \cos \theta + \sin \theta \cos(2\theta + \alpha))$$

$$\Rightarrow \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) = k \sin (3\theta + \alpha)$$

$$\Rightarrow \sin 2\theta = 2k \sin (3\theta + \alpha) \quad \dots(2)$$

From (1) and (2), we get

$$\tan 2\theta = 2 \tan (3\theta + \alpha)$$

Archives

JEE MAIN

Single Correct Answer Type

1. (3) $\cos (\beta - \gamma) + \cos (\gamma - \alpha) + \cos (\alpha - \beta) = -\frac{3}{2}$

$$\Rightarrow 2 [\cos (\beta - \gamma) + \cos (\gamma - \alpha) + \cos (\alpha - \beta)] + 3 = 0$$

$$\Rightarrow 2 [\cos (\beta - \gamma) + \cos (\gamma - \alpha) + \cos (\alpha - \beta)] + \sin^2 \alpha + \cos^2 \alpha + \sin^2 \beta + \cos^2 \beta + \sin^2 \gamma + \cos^2 \gamma = 0$$

$$\Rightarrow (\sin \alpha + \sin \beta + \sin \gamma)^2 + (\cos \alpha + \cos \beta + \cos \gamma)^2 = 0$$

$$\Rightarrow \cos \alpha + \cos \beta + \cos \gamma = 0$$

and $\sin \alpha + \sin \beta + \sin \gamma = 0$

2. (3) $\cos (\alpha + \beta) = \frac{4}{5}$

$$\Rightarrow \tan (\alpha + \beta) = \frac{3}{4}$$

$$\sin (\alpha - \beta) = \frac{5}{13}$$

$$\Rightarrow \tan (\alpha - \beta) = \frac{5}{12}$$

$$\therefore \tan 2\alpha = \tan (\alpha + \beta + \alpha - \beta)$$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33}$$

3. (2) $A = \sin^2 x + \cos^4 x$

$$= 1 - \cos^2 x + \cos^4 x$$

$$= 1 - \cos^2 x (1 - \cos^2 x)$$

$$= 1 - \cos^2 x \sin^2 x$$

$$= 1 - \frac{\sin^2 2x}{4}$$

Now, $0 \leq \sin^2 2x \leq 1$

$$\Rightarrow -\frac{1}{4} \leq -\frac{\sin^2 2x}{4} \leq 0$$

$$\Rightarrow \frac{3}{4} \leq 1 - \frac{\sin^2 2x}{4} \leq 1$$

$$\Rightarrow 3/4 \leq A \leq 1$$

4. (2) $3 \sin P + 4 \cos Q = 6$ (i)

$$4 \sin Q + 3 \cos P = 1$$
 (ii)

Squaring and adding (i) and (ii) we get $\sin (P + Q) = \frac{1}{2}$

$$\Rightarrow P + Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\Rightarrow R = \frac{5\pi}{6} \text{ or } \frac{\pi}{6}$$

If $R = \frac{5\pi}{6}$ then $0 < P, Q < \frac{\pi}{6}$

$$\Rightarrow \cos Q < 1 \text{ and } \sin P < \frac{1}{2}$$

$$\Rightarrow 3 \sin P + 4 \cos Q < \frac{11}{2}, \text{ which is not possible}$$

So $R = \frac{\pi}{6}$

5. (1) $5(\tan^2 x - \cos^2 x) = 2 \cos 2x + 9$

$$\Rightarrow 5(\sec^2 x - 1 - \cos^2 x) = 2(2 \cos^2 x - 1) + 9$$

$$\Rightarrow 5\left[\frac{1}{y} - 1 - y\right] = 2(2y - 1) + 9 \text{ (Putting } \cos^2 x = y)$$

$$\Rightarrow 9y^2 + 12y - 5 = 0$$

$$\Rightarrow (3y - 1)(3y + 5) = 0$$

$$\Rightarrow y = \frac{1}{3} \quad \left(\because y = -\frac{5}{3} \text{ is not possible}\right)$$

Now, $\cos 2x = 2 \cos^2 x - 1 = 2\left(\frac{1}{3}\right) - 1 = -\frac{1}{3}$

$$\cos 4x = 2 \cos^2 2x - 1 = 2\left(-\frac{1}{3}\right)^2 - 1 = -\frac{7}{9}$$

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Single Correct Answer Type

1. (3) $2 \sum_{k=1}^{13} \frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$

$$= 2 \sum \frac{\sin\left\{\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\right\}}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \cdot \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

$$= 2 \sum_{k=1}^{13} \left(\cot\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) \right)$$

$$= 2 \left[\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4} + \frac{13\pi}{6}\right) \right]$$

$$= 2[1 - (2 - \sqrt{3})]$$

$$= 2(\sqrt{3} - 1)$$

Multiple Correct Answers Type

1. (1), (2)

For $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Let $\cos 4\theta = \frac{1}{3}$

$$\Rightarrow \cos 2\theta = \pm \sqrt{\frac{1 + \cos 4\theta}{2}} = \pm \sqrt{\frac{2}{3}}$$

$$f\left(\frac{1}{3}\right) = \frac{2}{2 - \sec^2 \theta} = \frac{2 \cos^2 \theta}{2 \cos^2 \theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$$\Rightarrow f\left(\frac{1}{3}\right) = 1 - \sqrt{\frac{3}{2}} \text{ or } 1 + \sqrt{\frac{3}{2}}$$

2. (1), (3)

$$\cos \beta (2 + \cos \alpha) = 1 + 2 \cos \alpha$$

$$\Rightarrow \frac{\cos \beta}{1} = \frac{1 + 2 \cos \alpha}{2 + \cos \alpha}$$

$$\Rightarrow \frac{1 - \cos \beta}{1 + \cos \beta} = \frac{1 - \cos \alpha}{3(1 + \cos \alpha)}$$

Numerical Value Type

$$1. (2) \frac{1}{4 \cos^2 \theta + 1 + \frac{3}{2} \sin 2\theta} = \frac{1}{2[1 + \cos 2\theta] + 1 + \frac{3}{2} \sin 2\theta}$$

$$= \frac{1}{2 \cos 2\theta + \frac{3}{2} \sin 2\theta + 3}$$

Now,

$$-\sqrt{2^2 + \left(\frac{3}{2}\right)^2} \leq 2 \cos 2\theta + \frac{3}{2} \sin 2\theta \leq \sqrt{2^2 + \left(\frac{3}{2}\right)^2}$$

$$\text{or } -\frac{5}{2} \leq 2 \cos 2\theta + \frac{3}{2} \sin 2\theta \leq \frac{5}{2}$$

$$\Rightarrow \frac{1}{2} \leq 2 \cos 2\theta + \frac{3}{2} \sin 2\theta + 3 \leq \frac{11}{2}$$

$$\Rightarrow \frac{2}{11} \leq \frac{1}{2 \cos 2\theta + \frac{3}{2} \sin 2\theta + 3} \leq 2$$

Hence, the maximum value is 2.

$$2. (7) \frac{1}{\sin \frac{\pi}{n}} - \frac{1}{\sin \frac{3\pi}{n}} = \frac{1}{\sin \frac{2\pi}{n}}$$

$$\text{or } \frac{\sin \frac{3\pi}{n} - \sin \frac{\pi}{n}}{\sin \frac{\pi}{n} \sin \frac{3\pi}{n}} = \frac{1}{\sin \frac{2\pi}{n}}$$

$$\text{or } \frac{\left(2 \sin \frac{\pi}{n} \cos \frac{2\pi}{n}\right) \sin \frac{2\pi}{n}}{\sin \frac{\pi}{n} \sin \frac{3\pi}{n}} = 1$$

$$\text{or } \sin \frac{4\pi}{n} = \sin \frac{3\pi}{n}$$

$$\Rightarrow \frac{4\pi}{n} + \frac{3\pi}{n} = \pi$$

$$\Rightarrow n = 7.$$

$$\Rightarrow \tan^2 \frac{\alpha}{2} = 3 \tan^2 \frac{\beta}{2}$$

$$\Rightarrow \tan \frac{\alpha}{2} = \pm \sqrt{3} \tan \frac{\beta}{2}$$

Matrix Match Type

$$(q) \cos x + \cos y + \cos z = 0 \text{ and } \sin x + \sin y + \sin z = 0$$

$$\cos x + \cos y = -\cos z \quad \dots(1)$$

$$\text{and } \sin x + \sin y = -\sin z \quad \dots(2)$$

Squaring and adding we get,

$$1 + 1 + 2(\cos x \cos y + \sin x \sin y) = 1$$

$$\Rightarrow 2 + 2 \cos(x-y) = 1$$

$$\Rightarrow 2 \cos(x-y) = -1$$

$$\Rightarrow \cos(x-y) = -\frac{1}{2}$$

$$\Rightarrow 2 \cos^2 \left(\frac{x-y}{2} \right) - 1 = -\frac{1}{2}$$

$$\Rightarrow \cos \left(\frac{x-y}{2} \right) = \frac{1}{2}$$

$$(r) \cos \left(\frac{\pi}{4} - x \right) \cos 2x + \sin x \sin 2x \sec x$$

$$= \cos x \sin 2x \sec x + \cos \left(\frac{\pi}{4} + x \right) \cos 2x$$

$$\Rightarrow \left[\cos \left(\frac{\pi}{4} - x \right) - \cos \left(\frac{\pi}{4} + x \right) \right] \cos 2x$$

$$= (\cos x \sin 2x \sec x - \sin x \sin 2x \sec x)$$

$$\Rightarrow \frac{2}{\sqrt{2}} \sin x \cos 2x = (\cos x - \sin x) \sin 2x \sec x$$

$$\Rightarrow \sqrt{2} \sin x \cos 2x = (\cos x - \sin x) 2 \sin x$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{1}{\cos x + \sin x}$$

$$\Rightarrow x = \frac{\pi}{4}$$

$$\Rightarrow \sec x = \sec \frac{\pi}{4} = \sqrt{2}$$

Note: Solutions of the remaining parts are given in their respective chapters.

Chapter 4

Concept Application Exercises

Exercise 4.1

- $\sin^2 \theta - \cos \theta = \frac{1}{4}$
 or $1 - \cos^2 \theta - \cos \theta = \frac{1}{4}$
 or $4 \cos^2 \theta + 4 \cos \theta - 3 = 0$
 or $(2 \cos \theta + 3)(2 \cos \theta - 1) = 0$
 or $\cos \theta = \frac{1}{2}, -\frac{3}{2}$ (rejected)
 $\Rightarrow \theta = \pi/3, \frac{5\pi}{3}$ in the given interval
- $\cos^7 x = 1 - \sin^4 x = (1 - \sin^2 x)(1 + \sin^2 x)$
 or $\cos^7 x = \cos^2 x (2 - \cos^2 x)$
 or $\cos^2 x (\cos^5 x + \cos^2 x - 2) = 0$
 or $\cos x = 0$
 $\Rightarrow x = \pm\pi/2$ in the given interval
 or $\cos^5 x + \cos^2 x - 2 = 0$,
 which holds only when $\cos x = 1$; hence $x = 0$.
 Thus, there are total three solutions: $0, \pm\pi/2$.
- $(1 - 2 \cos \theta)^2 + (\tan \theta + \sqrt{3})^2 = 0$
 $\Rightarrow (1 - 2 \cos \theta)^2 = 0$
 and $(\tan \theta + \sqrt{3})^2 = 0$
 $\Rightarrow \cos \theta = \frac{1}{2}$ and $\tan \theta = -\sqrt{3}$
 Hence, θ is in the fourth quadrant, and
 $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$
 Hence, the general solution is
 $\theta = 2n\pi + \frac{5\pi}{3}, n \in \mathbb{Z}$.
- $\sin 3\theta - \sin \theta = 4 \cos^2 \theta - 2$
 $\Rightarrow 2 \sin \theta \cos 2\theta = 2 \cos 2\theta$
 $\Rightarrow \cos 2\theta = 0$ or $\sin \theta = 1$
 $\Rightarrow 2\theta = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$ or $\theta = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
 or $\theta = (2n+1)\frac{\pi}{4}, n \in \mathbb{Z}$, or $\theta = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
- If $\sin x > 0$, then we have
 $2\sin^2 x + \sin x - 1 = 0$
 $\Rightarrow \sin x = \frac{1}{2}$,
 $\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$
 If $\sin x < 0$, then we have
 $2\sin^2 x - \sin x - 1 = 0$
 $\Rightarrow \sin x = -\frac{1}{2}$
 $\therefore x = -\frac{\pi}{6}$

$$\begin{aligned}
 6. \quad & \sin^4 x + \cos^4 x - 2\sin^2 x + \frac{3}{4} \sin^2 2x = 0 \\
 \Rightarrow & (\sin^2 + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x - 2 \sin^2 x \\
 & + \frac{3}{4} \cdot 4 \sin^2 x \cdot \cos^2 x = 0 \\
 \Rightarrow & 1 - 2 \sin^2 x + \sin^2 x \cos^2 x = 0 \\
 \Rightarrow & \sin^4 x + \sin^2 x - 1 = 0 \\
 \Rightarrow & \sin^2 x = \frac{\sqrt{5}-1}{2} \\
 \Rightarrow & \sin x = \pm \sqrt{\frac{\sqrt{5}-1}{2}}
 \end{aligned}$$

There are two values of x for $\sin x = \sqrt{\frac{\sqrt{5}-1}{2}}$ and two values of x for $\sin x = -\sqrt{\frac{\sqrt{5}-1}{2}}$.

So there are four solutions.

$$\begin{aligned}
 7. \quad & \text{We have} \\
 & 2\sin x + 5\sin^2 x + 8\sin^3 x + \dots = 1 \\
 \therefore & 2\sin^2 x + 5\sin^3 x + \dots = \sin x \\
 & \text{Subtracting (2) from (1), we get} \\
 & 2\sin x + 3(\sin^2 x + \sin^3 x + \dots) = 1 - \sin x \\
 \therefore & 2\sin x + \frac{3\sin^2 x}{1 - \sin x} = 1 - \sin x \\
 \Rightarrow & 2\sin x - 2\sin^2 x + 3\sin^2 x = 1 - 2\sin x + \sin^2 x \\
 \Rightarrow & \sin x = \frac{1}{4}
 \end{aligned}$$

So, there will be two solutions.

$$\begin{aligned}
 8. \quad & \text{We have,} \\
 & \cos x + \cos y = \frac{3}{2} \\
 \therefore & 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) = \frac{3}{2} \\
 \Rightarrow & 2 \cos \frac{\pi}{3} \cos \left(\frac{x-y}{2} \right) = \frac{3}{2} \\
 \Rightarrow & \cos \left(\frac{x-y}{2} \right) = \frac{3}{2}, \text{ which is not possible.}
 \end{aligned}$$

Hence, the system of equations has no solution.

$$\begin{aligned}
 9. \quad & \operatorname{cosec}^2 \theta - \cot^2 \theta = \cos \theta \\
 \Rightarrow & \cos \theta = 1 \\
 \Rightarrow & \sin \theta = 0, \text{ which is not possible as } \operatorname{cosec} \theta \text{ and } \cot \theta \text{ are not defined for this value of } \sin \theta.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \sin x \tan x - \sin x + \tan x - 1 = 0 \\
 \Rightarrow & \tan x (\sin x + 1) - (\sin x + 1) = 0 \\
 \Rightarrow & (\sin x + 1)(\tan x - 1) = 0 \\
 \Rightarrow & \sin x = -1 \\
 \text{or} \quad & \tan x = 1 \\
 & \text{For } \sin x = -1, \tan x \text{ is not defined (as } \cos x = 0). \\
 \therefore & \tan x = 1 \\
 \therefore & x = \frac{\pi}{4}, \frac{5\pi}{4}
 \end{aligned}$$

10. We have $\sin^2 \theta - \frac{4}{\sin^3 \theta - 1} = 1 - \frac{4}{\sin^3 \theta - 1}$, where $\sin \theta \neq 1$

$\therefore \sin^2 \theta = 1, \sin \theta \neq 1$
 $\therefore \sin \theta = -1$

Thus, there are three values of θ one each in intervals $[0, 2\pi]$, $[2\pi, 4\pi]$ and $[4\pi, 6\pi]$.

11. $\log_{\sin x} (1 + \cos x) = 2$
 $\Rightarrow 1 + \cos x = \sin^2 x$

$\Rightarrow 1 + \cos x = (1 - \cos x)(1 + \cos x)$

$\Rightarrow 1 + \cos x = 0$ or $1 - \cos x = 1$

$\Rightarrow \cos x = -1$ or $\cos x = 0$

$\Rightarrow \sin x = 0$ or $|\sin x| = 1$

Both of these are not possible as base of logarithm cannot be either 0 or 1.

Exercise 4.2

1. $2 \sin \theta + 1 = 0$

$\Rightarrow \sin \theta = -\frac{1}{2} = \sin \left(\frac{-\pi}{6} \right)$

$\Rightarrow \theta = n\pi + (-1)^n \left(\frac{-\pi}{6} \right), n \in \mathbb{Z}$

$= n\pi + (-1)^{n+1} \frac{\pi}{6}, n \in \mathbb{Z}$

2. $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2 \theta$

or $\sin(n\theta - (n-1)\theta) \sin(n\theta + (n-1)\theta) = \sin^2 \theta$

or $\sin \theta \sin((2n-1)\theta) = \sin^2 \theta$

or $\sin \theta = 0$ or $\sin((2n-1)\theta) = \sin \theta$

$\Rightarrow \theta = k\pi$, or $(2n-1)\theta = m\pi + (-1)^m \theta; k, m \in \mathbb{Z}$

$\Rightarrow \theta = k\pi$, or $(2n-1)\theta = m\pi + \theta$, when m is even

and $(2n-1)\theta = m\pi - \theta$, when m is odd

$\Rightarrow \theta = k\pi$, or $\theta = m\pi/(2n-2)$, when m is even and

$\theta = m\pi/2n$, when m is odd

$\Rightarrow \theta = k\pi$, or $\theta = p\pi/(n-1)$, and $\theta = (2q+1)\pi/2n$,

where $k, p, n, q \in \mathbb{Z}$

3. $(\cos \theta + \cos 7\theta) + (\cos 3\theta + \cos 5\theta) = 0$

or $2 \cos 4\theta (\cos 3\theta + \cos \theta) = 0$

or $4 \cos 4\theta \cos 2\theta \cos \theta = 0$

or $4 \times \frac{1}{2^3 \sin \theta} (\sin 2^3 \theta) = 0$

$\Rightarrow \sin 8\theta = 0$ or $\theta = n\pi/8, n \in \mathbb{Z}$

4. $\tan^2 \theta - 2 \sin \theta = 0$

or $3 \frac{\sin^2 \theta}{\cos^2 \theta} - 2 \sin \theta = 0, \cos \theta \neq 0$

or $3 \sin^2 \theta - 2 \sin \theta (1 - \sin^2 \theta) = 0$

or $3 \sin^2 \theta - 2 \sin \theta (1 - \sin^2 \theta) = 0$

or $\sin \theta (2 \sin^2 \theta + 3 \sin \theta - 2) = 0$

or $\sin \theta (2 \sin \theta - 1) (\sin \theta + 2) = 0$

$\sin \theta = 0, 1 - 2$ (rejected)

$\Rightarrow \theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}$

5. $1^2 = \sin \theta \cos 2\theta$

or $1 - \sin \theta (1 - 2 \sin^2 \theta) = 0$

or $2 \sin^3 \theta - \sin \theta + 1 = 0$

or $(\sin \theta + 1)(2 \sin^2 \theta - 2 \sin \theta + 1) = 0$
 or $\sin \theta = -1$

The other factor gives imaginary roots

$\Rightarrow \theta = n\pi + (-1)^n \left(\frac{-\pi}{2} \right)$

$= n\pi - (-1)^n \frac{\pi}{2} = n\pi + (-1)^{n-1} \frac{\pi}{2}$

6. R.H.S. = 1

$\Rightarrow \tan x + \tan 3x = 0$

$\Rightarrow \sin 4x = 0$

$\Rightarrow x = \frac{n\pi}{4}, n \in \mathbb{Z}$

$\Rightarrow x = \pi/4, 3\pi/4, \pi$

Exercise 4.3

1. $\cos \theta = 1/3$ or $\cos \theta = \cos^{-1}(\cos 1/3)$

$\Rightarrow \theta = 2n\pi \pm \cos^{-1} \frac{1}{3}, n \in \mathbb{Z}$

2. The given equation can be written as

$\cos 4\theta \cos \theta - \sin 4\theta \sin \theta = 0$

$\Rightarrow \cos 5\theta = 0$

$\Rightarrow 5\theta = \left(n + \frac{1}{2} \right) \pi, n \in \mathbb{Z}$

$\Rightarrow \theta = \frac{(2n+1)\pi}{10}, 0 < \theta < \pi, n \in \mathbb{Z}$

$= \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Thus, there are only five solutions in the given range.

3. $\frac{\cos(x/2)}{\sin(x/2)} - \frac{\cos x}{2 \sin(x/2) \cos(x/2)} = \frac{1}{\sin(x/2)}$

$\Rightarrow 2 \cos^2(x/2) - \cos(x) = 2 \cos(x/2)$

$\Rightarrow 1 + \cos x - \cos x = 2 \cos(x/2)$

$\Rightarrow \cos(x/2) = 1/2 = \cos(\pi/3)$

$\Rightarrow x/2 = 2n\pi \pm \pi/3, n \in \mathbb{Z}$

$\Rightarrow x = 4n\pi \pm 2\pi/3, n \in \mathbb{Z}$

4. $\cot \theta + \tan \theta = 2 \operatorname{cosec} \theta$

or $\frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} = \frac{2}{\sin \theta}$

or $\sin \theta = 2 \sin \theta \cos \theta$

or $1 = 2 \cos \theta$

or $\cos \theta = 1/2$

$\Rightarrow \theta = 2n\pi \pm \pi/3, n \in \mathbb{Z}$

5. $\sin 6\theta = \sin 4\theta - \sin 2\theta$

or $\sin 6\theta + \sin 2\theta = \sin 4\theta$

or $2 \sin 4\theta \cos 2\theta = \sin 4\theta$

or $\sin 4\theta = 0$ or $\cos 2\theta = 1/2$

$\Rightarrow 4\theta = n\pi$ or $2\theta = 2m\pi \pm \pi/3; m, n \in \mathbb{Z}$

or $\theta = n\pi/4$ or $\theta = m\pi \pm \pi/6; m, n \in \mathbb{Z}$

6. $\cos \theta + \cos 2\theta + \cos 3\theta = 0$

or $(\cos \theta + \cos 3\theta) + \cos 2\theta = 0$

or $2 \cos \theta \cos 2\theta + \cos 2\theta = 0$

or $\cos 2\theta (2 \cos \theta + 1) = 0$

[$\because \sin \theta \neq 0$]

$$\begin{aligned} \text{or } \cos 2\theta &= 0 \text{ or } 2 \cos \theta + 1 = 0 \\ \Rightarrow 2\theta &= (2n+1)\pi/2 \text{ or } \cos \theta = -1/2, n \in \mathbb{Z} \\ \text{or } \theta &= (2n+1)\pi/4 \text{ or } \theta = 2n\pi \pm 2\pi/3, \forall n \in \mathbb{Z} \end{aligned}$$

$$7. \sqrt{1 + \sin 2x} - \sqrt{2} \cos 3x = 0$$

$$\text{or } \sqrt{1 + \sin 2x} = \sqrt{2} \cos 3x$$

$$\text{or } 1 + \sin 2x = 2 \cos^2 3x$$

$$1 + \sin 2x = 1 + \cos 6x$$

$$\text{or } \cos \left(\frac{\pi}{2} - 2x \right) = \cos 6x$$

$$\text{or } \frac{\pi}{2} - 2x = 2n\pi \pm 6x$$

$$\therefore x = \frac{n\pi}{2} - \frac{\pi}{8} \text{ or } x = \frac{n\pi}{4} + \frac{\pi}{16}$$

We get smallest value when $n = 0$ then $x = \pi/16$.

$$8. \cos p\theta = -\cos q\theta = \cos(\pi - q\theta)$$

$$\Rightarrow p\theta = 2n\pi \pm (\pi - q\theta)$$

$$\Rightarrow (p \pm q)\theta = (2n \pm 1)\pi$$

$$\Rightarrow \theta = \frac{(2n \pm 1)\pi}{(p \mp q)}, n \in \mathbb{Z}$$

$$\Rightarrow \theta = \frac{r\pi}{p \pm q}, \text{ where } r = -3, -1, 1, 3, \dots$$

$$\Rightarrow \theta = \dots, \frac{-3\pi}{p \pm q}, \frac{-\pi}{p \pm q}, \frac{\pi}{p \pm q}, \frac{3\pi}{p \pm q}, \dots$$

The above series is an A.P. of common difference $\frac{2\pi}{p \pm q}$.

$$9. \sin 3x + (\sin 5x + \sin x) = 0$$

$$\Rightarrow \sin 3x + (2 \sin 3x \cos 2x) = 0$$

$$\Rightarrow \sin 3x = 0$$

$$\text{or } \cos 2x = -\frac{1}{2} = \cos \frac{2\pi}{3}, n \in \mathbb{Z}$$

$$\Rightarrow x = n\pi/3$$

$$\text{or } x = 2n\pi \pm \frac{2\pi}{3}$$

Then $x = 0, \pi/3, 2\pi/3$,

Hence, there are three solutions.

Exercise 4.4

$$1. \text{ We have } \tan a\theta = \tan b\theta$$

$$\Rightarrow a\theta = n\pi + b\theta, n \in \mathbb{Z}$$

$$\Rightarrow (a - b)\theta = n\pi$$

$$\Rightarrow n\pi/(a - b) = \theta$$

Thus, the values of θ form an A.P. with common difference $\pi/(a - b)$.

$$2. \tan^2 \theta + 2\sqrt{3} \tan \theta = 1$$

$$\text{or } (\tan \theta + \sqrt{3})^2 = 4$$

$$\text{or } \tan \theta = 2 - \sqrt{3} \text{ or } -2 - \sqrt{3}$$

$$\text{or } \tan \theta = \tan 15^\circ, -\tan 75^\circ$$

$$= \tan \frac{\pi}{12}, \tan \left(-\frac{5\pi}{12} \right)$$

From $\tan \theta = \tan \frac{\pi}{12}$, we get

$$\theta = n\pi + \frac{\pi}{12}, n \in \mathbb{Z}$$

From $\tan \theta = \tan \left(-\frac{5\pi}{12} \right)$, we get

$$\theta = m\pi - \frac{5\pi}{12}, m \in \mathbb{Z}$$

$$3. \tan^2 x + \tan x - \sqrt{3} \tan x - \sqrt{3} = 0$$

$$\text{or } (\tan x - \sqrt{3})(\tan x + 1) = 0$$

$$\text{or } \tan x = \sqrt{3} \text{ or } \tan x = -1$$

$$\tan x = \sqrt{3} = \tan(\pi/3)$$

$$\Rightarrow x = n\pi + \pi/3, n \in \mathbb{Z}$$

$$\text{and } \tan x = -1 = \tan(-\pi/4)$$

$$\Rightarrow x = m\pi - \pi/4, m \in \mathbb{Z}$$

$$4. 3(\cos^2 \theta - \sin^2 \theta) = 2\sqrt{3} \sin \theta \cos \theta$$

$$\text{or } 3 \cos 2\theta = \sqrt{3} \sin 2\theta$$

$$\text{or } \tan 2\theta = \sqrt{3} = \tan(\pi/3)$$

$$\Rightarrow 2\theta = n\pi + \pi/3$$

$$\text{or } \theta = n\pi/2 + \pi/6, n \in \mathbb{Z}$$

$$5. \tan \theta + \tan(\theta + \pi/3) + \tan(\theta + 2\pi/3) = 3$$

$$\text{or } \tan \theta + \frac{\tan \theta + \tan(\pi/3)}{1 - \tan \theta \tan(\pi/3)} + \frac{\tan \theta + \tan(2\pi/3)}{1 - \tan \theta \tan(2\pi/3)} = 3$$

$$\text{or } \tan \theta + \frac{\tan \theta + \sqrt{3}}{1 - \sqrt{3} \tan \theta} + \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} = 3$$

$$\text{or } 3 \times \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} = 3$$

$$\text{or } \tan 3\theta = 1 = \tan(\pi/4), n \in \mathbb{Z}$$

$$\Rightarrow 3\theta = n\pi + \pi/4, n \in \mathbb{Z}$$

$$\text{or } \theta = (4n + 1)\pi/12, \forall n \in \mathbb{Z}$$

$$6. \text{ Since } \sin x = 0 \text{ does not satisfy the equation, dividing the equation by } \sin^3 x, \text{ we get}$$

$$\cot x \operatorname{cosec}^2 x = 2$$

$$\text{or } \cot^3 x + \cot x - 2 = 0$$

$$\text{or } (\cot x - 1)(\cot^2 x + \cot x + 2) = 0$$

$$\text{or } \cot x = 1$$

$$\Rightarrow x = n\pi + \pi/4, n \in \mathbb{Z}$$

$$7. 7 \cos^2 x + \sin x \cos x - 3 = 0$$

Dividing by $\cos^2 x$, we get

$$7 + \tan x - 3 \sec^2 x = 0$$

$$\text{or } 7 + \tan x - 3(1 + \tan^2 x) = 0$$

$$\text{or } 3 \tan^2 x - \tan x - 4 = 0$$

$$\text{or } (\tan x + 1)(3 \tan x - 4) = 0$$

$$\text{or } \tan x = -1 \text{ and } \tan x = \frac{4}{3}$$

$$\Rightarrow x = n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}$$

$$\text{and } x = k\pi + \tan^{-1} \left(\frac{4}{3} \right), k \in \mathbb{Z}$$

$$8. \tan \left(\frac{p\pi}{4} \right) = \cot \left(\frac{q\pi}{4} \right) = \tan \left(\frac{\pi}{2} - \frac{q\pi}{4} \right)$$

$$\Rightarrow \frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4}$$

$$\text{or } \frac{p}{4} = n + \frac{1}{2} - \frac{q}{4}$$

$$\text{or } \frac{p+q}{4} = \frac{2n+1}{2}$$

$$\text{or } p+q = 2(2n+1), n \in \mathbb{Z}$$

$$9. \sec \theta - 1 = (\sqrt{2} - 1) \tan \theta$$

$$\Rightarrow \frac{(1 - \cos \theta)}{\cos \theta} = \frac{(\sqrt{2} - 1) \sin \theta}{\cos \theta}$$

$$\text{or } 2 \sin^2 (\theta/2) = (\sqrt{2} - 1) 2 \sin (\theta/2) \cos (\theta/2)$$

$$\text{or } \sin (\theta/2) = 0, \tan (\theta/2) = (\sqrt{2} - 1) = \tan (\pi/8)$$

$$\Rightarrow \theta/2 = n\pi \quad \text{or} \quad \frac{\theta}{2} = n\pi + \frac{\pi}{8}, n \in \mathbb{Z}$$

$$\Rightarrow \theta = 2n\pi \quad \text{or} \quad \theta = 2n\pi + \frac{\pi}{4}$$

Exercise 4.5

$$1. (\tan^2 \theta - 1)^2 = 0$$

$$\text{or } \tan^2 \theta = 1$$

$$\Rightarrow \theta = n\pi \pm \pi/4, n \in \mathbb{Z}$$

$$2. \sec^2 \theta + \tan^2 \theta = \frac{5}{3}$$

$$\text{or } \tan^2 \theta = \frac{1}{3} = \tan^2 \left(\frac{\pi}{6} \right)$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

$$3. 4 + 2 \sin^2 x = 5$$

$$\text{or } \sin^2 x = \frac{1}{2} = \sin^2 \frac{\pi}{4}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

$$4. 2^{\cos 2x} + 1 = 3 \cdot 2^{-\sin^2 x}$$

$$\Rightarrow 2^{1-2\sin^2 x} + 1 = 3 \cdot 2^{-\sin^2 x}$$

$$\text{Let } 2^{-\sin^2 x} = t$$

Now, given equation reduces to

$$2t^2 + 1 = 3t$$

$$\Rightarrow t = 1 \text{ and } t = 1/2$$

If $t = 1$ then $\sin^2 x = 0$

$$\Rightarrow x = n\pi, n \in \mathbb{I}$$

$$\text{If } t = \frac{1}{2} \text{ then } \sin^2 x = 1$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{2}, n \in \mathbb{I}$$

5. We have

$$\cot^2(\sin x + 3) = 1 = \cot^2 \frac{\pi}{4}$$

$$\Rightarrow \sin x + 3 = n\pi \pm \frac{\pi}{4}$$

$$\text{Now, } 2 \leq \sin x + 3 \leq 4$$

$$\therefore \sin x + 3 = \pi - \frac{\pi}{4}$$

$$\text{or } \sin x + 3 = \pi + \frac{\pi}{4}$$

$$\therefore \sin x = \frac{3\pi}{4} - 3$$

$$\text{or } \sin x = \frac{5\pi}{4} - 3$$

...(1)

...(2)

For (1), we have two values of x in interval $(\pi, 2\pi)$.

For (2), we have two values of x in each of the intervals $(0, \pi)$, $(2\pi, 3\pi)$.

So, there are total six solutions.

Exercise 4.6

$$1. \cot \theta + \operatorname{cosec} \theta = \sqrt{3}$$

$$\text{or } \frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta} = \sqrt{3}$$

$$\text{or } \cos \theta + 1 = \sqrt{3} \sin \theta$$

$$\text{or } \sqrt{3} \sin \theta - \cos \theta = 1$$

...(i)

$$\text{or } 2 \cos \left(\theta + \frac{\pi}{3} \right) = -1$$

$$\text{or } \cos \left(\theta + \frac{\pi}{3} \right) = -\frac{1}{2}$$

$$\text{or } \cos \left(\theta + \frac{\pi}{3} \right) = \cos \frac{2\pi}{3}$$

$$\Rightarrow \theta + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$$

$$\text{or } \theta = 2n\pi \pm \frac{2\pi}{3} - \frac{\pi}{3}, n \in \mathbb{Z}$$

$$\therefore \theta = 2n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$$

$$\text{or } \theta = 2n\pi - \pi = (2n-1)\pi, n \in \mathbb{Z}$$

But θ cannot be equal to $(2n-1)\pi$ as it makes $\sin \theta = 0$.

$$\text{Hence, } \theta = 2n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$$

$$2. \sin \theta + \cos \theta = \sqrt{2} \cos A$$

$$\Rightarrow (1/\sqrt{2}) \sin \theta + (1/\sqrt{2}) \cos \theta = \cos A$$

$$\text{or } \cos (\theta - \pi/4) = \cos A$$

$$\Rightarrow \theta - \pi/4 = 2n\pi \pm A$$

$$\text{or } \theta = 2n\pi + \pi/4 \pm A, \forall n \in \mathbb{Z}$$

$$3. (\sqrt{2}/\cos \theta) + \sin \theta / \cos \theta = 1$$

($\because \cos \theta \neq 0$)

$$\text{or } \cos \theta - \sin \theta = \sqrt{2}$$

$$\Rightarrow (1/\sqrt{2}) \cos \theta - (1/\sqrt{2}) \sin \theta = 1$$

$$\Rightarrow \cos (\theta + \pi/4) = 1 = \cos 0$$

$$\Rightarrow \theta + (\pi/4) = 2n\pi$$

$$\Rightarrow \theta = 2n\pi - (\pi/4), n \in \mathbb{Z}$$

$$4. \text{ We know that } a \cos \theta + b \sin \theta = c \text{ has solution only when}$$

$$|c| \leq \sqrt{a^2 + b^2}.$$

Then for the given equation, we must have

$$|2k+1| \leq \sqrt{74}$$

$$\Rightarrow -\sqrt{74} < 2k+1 < \sqrt{74}$$

$$\Rightarrow -8 < 2k+1 < 8 \text{ (for integral solutions)}$$

$$\Rightarrow k = -4, -3, -2, -1, 0, 1, 2, 3$$

Thus, eight values of k will satisfy the given inequality.

Exercise 4.7

$$1. \text{ We have } \cos x + \cos 2x + \dots + \cos (nx) = n$$

$$\text{Now, } \cos x + \cos 2x + \dots + \cos (nx) \leq n$$

$$\text{So, } \cos x = \cos 2x = \dots = \cos nx = 1$$

$$\therefore x = 0$$

2. The given equation is

$$1 + \sin^2 ax = \cos x \quad \dots(i)$$

$$\text{or } (1 - \cos x) + \sin^2 ax = 0$$

$$\text{or } 2 \sin^2 (x/2) + \sin^2 ax = 0$$

Since both the terms on the L.H.S. are non-negative, their sum will be zero only if each one is zero. Thus,

$$\sin (x/2) = 0 \text{ and } \sin ax = 0$$

$$\Rightarrow x/2 = n\pi \text{ and } ax = m\pi, \forall n, m \in \mathbb{Z}$$

$$\Rightarrow x = 2n\pi \text{ and } x = m\pi/a, \forall n, m \in \mathbb{Z}$$

$$\Rightarrow 2n\pi = m\pi/a \Rightarrow 2an = m \quad \dots(ii)$$

Since m and n are integer and a is irrational, Eq. (ii) is possible only if $m = 0$ and $n = 0$.

Hence, $x = 0$ is the only solution.

3. $\sin^4 x = 1 + \tan^8 x$

Now, $1 + \tan^8 x \geq 1$ and $\sin^4 x \leq 1$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence, for the given equation to be satisfied,

$$\sin^4 x = 1 \text{ and } 1 + \tan^8 x = 1$$

$$\Rightarrow \sin^2 x = 1 \text{ and } \tan^8 x = 0$$

which is never possible, since $\sin x$ and $\tan x$ vanish simultaneously.

Therefore, the given equation has no solution.

$$4. \sin x \left(\cos \frac{x}{4} - 2 \sin x \right) + \left(1 + \sin \frac{x}{4} - 2 \cos x \right) \cos x = 0$$

$$\text{or } \left(\sin x \cos \frac{x}{4} + \sin \frac{x}{4} \cos x \right) - 2(\sin^2 x + \cos^2 x) + \cos x = 0$$

$$\text{or } \sin \frac{5x}{4} + \cos x = 2 \quad \dots(i)$$

Now Eq. (i) will hold if

$$\sin \frac{5x}{4} = 1 \text{ and } \cos x = 1$$

$$\Rightarrow \frac{5x}{4} = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z} \text{ and } x = 2m\pi, m \in \mathbb{Z}$$

$$\Rightarrow x = \frac{(8n+2)\pi}{5}, n \in \mathbb{Z} \quad \dots(ii)$$

$$\text{and } x = 2m\pi, m \in \mathbb{Z} \quad \dots(iii)$$

Comparing the values of x , we get

$$\frac{(8n+2)\pi}{5} = 2m\pi$$

$$\text{or } 8n+2 = 10m$$

$$\text{or } n = \frac{5m-1}{4}$$

Value of m	Value of n
1	1
5	6
9	11
...	...
...	...
$4k-3$	$5k-4$

Therefore, the general solution of the given equation can be obtained by substituting either $m = 4k-3$ in Eq. (iii) or $n = 5p-4$ in Eq. (ii).

Therefore, the general solution of Eq. (i) is

$$x = (8k-6)\pi, k \in \mathbb{Z}.$$

$$5. 12 \sin x + 5 \cos x = 2y^2 - 8y + 21$$

$$\Rightarrow \sqrt{12^2 + 5^2} \left(\frac{12}{13} \sin x + \frac{5}{13} \cos x \right) = 2(y^2 - 4y + 4) + 13$$

$$\Rightarrow 13 \cos(x - \alpha) = 2(y-2)^2 + 13$$

where $\cos \alpha = 5/13$.

Now L.H.S. ≤ 13 and R.H.S. ≥ 13

$$\Rightarrow 13 \cos(x - \alpha) = 2(y-2)^2 + 13 = 13$$

$$\Rightarrow \cos(x - \alpha) = 1 \text{ and } y = 2$$

$$\Rightarrow x - \alpha = 2n\pi \text{ and } y = 2$$

$$\Rightarrow x = 2n\pi + \cos^{-1} \left(\frac{5}{13} \right) \text{ and } y = 2, \text{ where } n \in \mathbb{Z}$$

$$6. \sin 2x + \cos 4x = 2$$

It is possible only when $\sin 2x = 1$ and $\cos 4x = 1$

$$2x = 2n\pi + \frac{\pi}{2} \text{ and } 4x = 2m\pi$$

$$\therefore x = n\pi + \frac{\pi}{4} \text{ and } x = \frac{m\pi}{2}, \text{ where } m, n \in \mathbb{Z}$$

Then, solution is given by

$$\left(n\pi + \frac{\pi}{4}, n \in \mathbb{Z} \right) \cap \left(\frac{m\pi}{2}, m \in \mathbb{Z} \right) = \emptyset$$

$$7. \tan(P \cot x) = \cot(P \tan x)$$

$$\text{or } \tan(P \cot x) = \tan \left(\frac{\pi}{2} - P \tan x \right)$$

$$\text{or } P \cot x = \frac{\pi}{2} - P \tan x$$

$$\text{or } P(\tan x + \cot x) = \frac{\pi}{2} \text{ where } P > 0$$

Now, $\tan x + \cot x \geq 2$

$$\text{or } 2P \leq P(\tan x + \cot x) = \frac{\pi}{2} \text{ or } P \leq \frac{\pi}{4}$$

$$8. \text{LHS} \geq 2; \text{RHS} \leq 2$$

Equality is possible.

Then for RHS, $x = 1$

$$\text{For LHS, } \tan^2 \pi(1+y) = 1$$

$$\therefore \tan^2 \pi y = 1$$

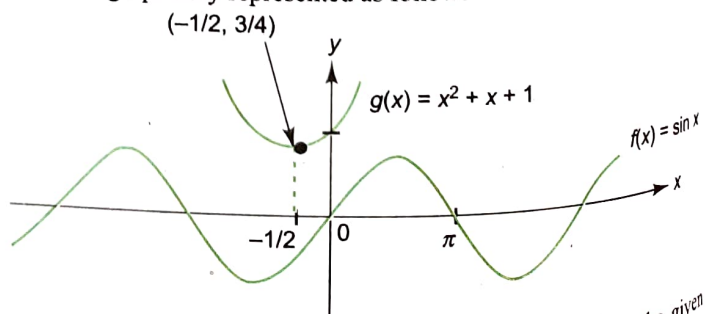
$$\text{Least value of } y \text{ is } \frac{1}{4}.$$

$$9. \cos^5 x \leq \cos^2 x \text{ and } \sin^3 x \leq \sin^2 x$$

So, $\cos^5 x + \sin^3 x = 1$ is possible only when $\cos^5 x = \cos^2 x$ and $\sin^3 x = \sin^2 x$, which is possible only at $x = 0, \pi/2$, and 2π .

Exercise 4.8

1. Let $f(x) = \sin x$ and $g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$, which could be graphically represented as follows:



As these two graphs do not intersect at any point, the given equation has no solution.

2. We have $\sin x = \log_e x$

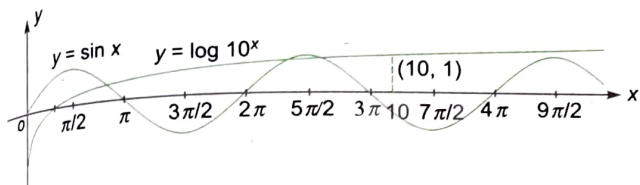
To find the number of roots of above equation, we draw the graphs of $y = \sin x$ and $y = \log_{10} x$ and count the number of points of intersection.

Now, $\sin x \in [-1, 1]$.

Further, we know that $y = \log_{10} x$ always increases.

So, $y = \log_{10} x$ will never intersect $y = \sin x$, when $\log_{10} x > 1$ or $x > 10$.

Graphs of both functions are as shown in the following figure.

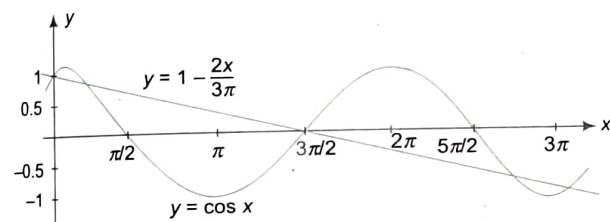


From the figure, we can see that graphs intersect at three points.

3. We have $2x = 3\pi(1 - \cos x)$

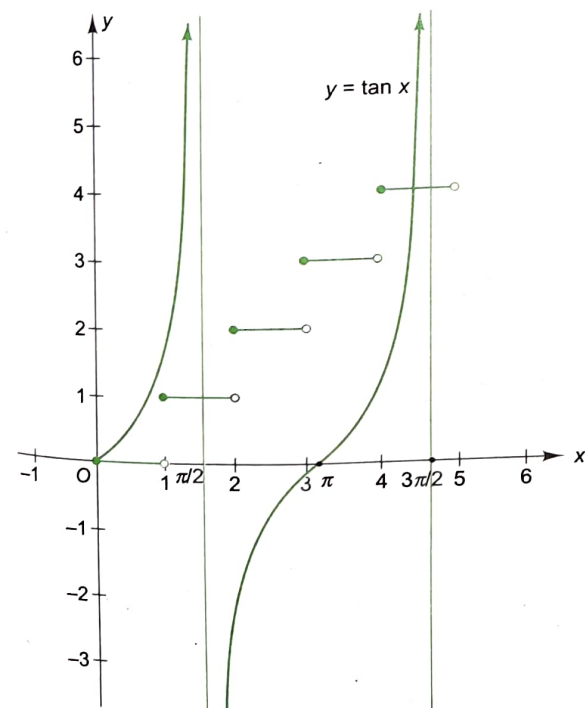
$$\text{or } \cos x = 1 - \frac{2x}{3\pi}$$

Draw the graphs of $y = \cos x$ and $y = 1 - \frac{2x}{3\pi}$



From the above figure number of solutions = 5

4. Graphs of $y = \tan x$ and $y = [x]$ are drawn as shown in the following figure.



Clearly, $\tan 1 > \tan(\pi/4)$ or $\tan 1 > 1$.

So, $y = \tan x$ and $y = [x]$ do not intersect for $x \in (0, \pi/2)$.

Further from the figure, graphs intersect when $[x] = 4$.

$$\therefore \tan x = 4 \text{ or } x = \tan^{-1} 4.$$

Exercise 4.9

1. $\sin^2 \theta > \cos^2 \theta$

$$\Rightarrow \cos 2\theta < 0$$

$$\Rightarrow (\pi/2) < 2\theta < (3\pi/2)$$

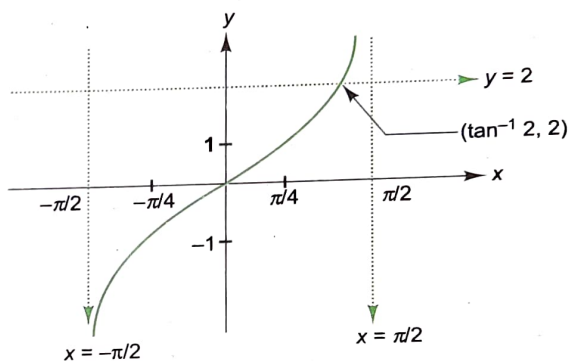
$$\text{or } (\pi/4) < \theta < (3\pi/4)$$

Taking general values, i.e., adding $2n\pi$, we get

$$2n\pi + \pi/2 < 2\theta < 2n\pi + 3\pi/2, n \in \mathbb{Z}$$

$$\text{or } n\pi + \pi/4 < \theta < n\pi + 3\pi/4$$

2. We know that $\tan x$ is periodic with period π . So, check the solution on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



It is clear from figure

$$\tan x < 2 \text{ when } -\frac{\pi}{2} < x < \tan^{-1} 2$$

General solution is

$$n\pi - \frac{\pi}{2} < x < n\pi + \tan^{-1} 2, n \in \mathbb{Z}$$

$$\Rightarrow x \in \left(n\pi - \frac{\pi}{2}, n\pi + \tan^{-1} 2\right), n \in \mathbb{Z}$$

3. $\sin 2x > \sqrt{2} \sin^2 x + (2 - \sqrt{2}) \cos^2 x$

$$\Rightarrow 2 \sin x \cos x > \sqrt{2} \sin^2 x + (2 - \sqrt{2}) \cos^2 x$$

$$\Rightarrow \tan^2 x - \sqrt{2} \tan x + (\sqrt{2} - 1) < 0$$

$$\Rightarrow (\tan x - 1)(\tan x - (\sqrt{2} - 1)) < 0$$

$$\Rightarrow (\sqrt{2} - 1) < \tan x < 1$$

$$\Rightarrow \pi/8 < x < \pi/4$$

$$\therefore x \in \left(n\pi + \frac{\pi}{8}, n\pi + \frac{\pi}{4}\right) \text{ where } n \in \mathbb{Z}.$$

4. $\tan^3 x - \tan^2 x + 3 - 3\tan x > 0$

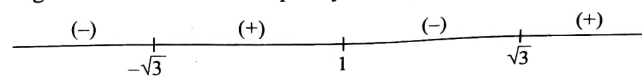
$$\Rightarrow \tan^2 x(\tan x - 1) - 3(\tan x - 1) > 0$$

$$\Rightarrow (\tan x - 1)(\tan^2 x - 3) > 0$$

$$\Rightarrow (\tan x - 1)(\tan x + \sqrt{3})(\tan x - \sqrt{3}) > 0$$

$$\Rightarrow (y - 1)(y + \sqrt{3})(y - \sqrt{3}) > 0, \text{ where } \tan x = y$$

Sign-scheme of above inequality is as follows:



$$\therefore -\sqrt{3} < y < 1 \text{ or } y > \sqrt{3}$$

$$\Rightarrow -\sqrt{3} < \tan x < 1 \text{ or } \tan x > \sqrt{3}$$

For $-\pi/2 < x < \pi/2$, we have $-\pi/3 < x < \pi/4$ or $\pi/3 < x < \pi/2$

$\therefore$ General solution is

$$x \in \left(n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{2} \right) \cup \left(n\pi - \frac{\pi}{3}, n\pi + \frac{\pi}{4} \right) \text{ where } n \in \mathbb{Z}.$$

5. We have $2 \cos^2 x + \sin x \leq 2$

$$\therefore 2(1 - \sin^2 x) + \sin x \leq 2$$

$$\Rightarrow -2 \sin^2 x + \sin x \leq 0$$

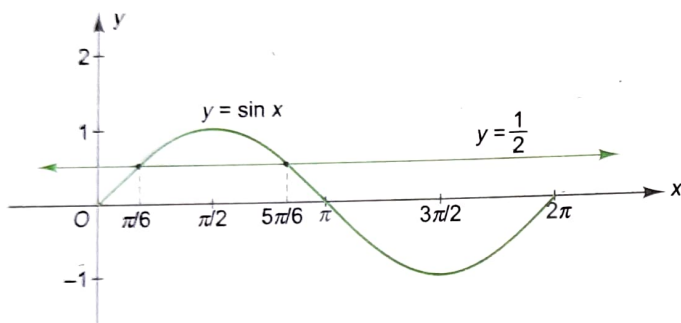
$$\Rightarrow 2 \sin^2 x - \sin x \geq 0$$

$$\Rightarrow \sin x (2 \sin x - 1) \geq 0$$

$$\Rightarrow \sin x (\sin x - 1/2) \geq 0$$

$$\Rightarrow \sin x \leq 0 \text{ or } \sin x \geq 1/2$$

Draw the graph of $y = \sin x$ and $y = 1/2$.



From the figure,

$$\sin x \geq 1/2$$

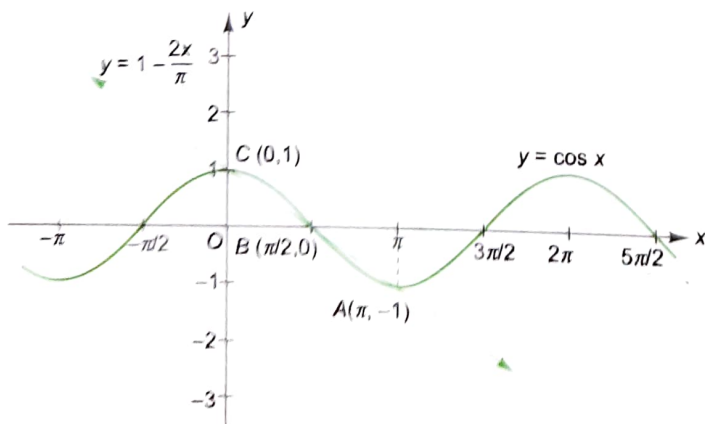
$$\Rightarrow \pi/2 \leq x \leq 5\pi/6$$

$$\text{and } \sin x \leq 0$$

$$\Rightarrow \pi \leq x \leq 3\pi/2$$

Hence the required solution is $x \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$.

6. Graphs of $y = \cos x$ and $y = 1 - \frac{2x}{\pi}$ are drawn as shown in the following figure.



From the figure,

$$\cos x > 1 - \frac{2x}{\pi} \text{ for } x \in (0, \pi/2) \cup (\pi, \infty)$$

Exercises

Single Correct Answer Type

1. (1) $\sin \theta = 1/2$ and $\cos \theta = -\sqrt{3}/2$

$\Rightarrow \theta$ lies in the second quadrant.

$$\Rightarrow \sin \theta = \sin 5\pi/6; \cos \theta = \cos 5\pi/6;$$

$$\therefore \theta = 2n\pi + (5\pi/6), n \in \mathbb{Z}$$

2. (3) Since $\tan \theta < 0$ and $\cos \theta > 0$, θ lies in the fourth quadrant. Then $\theta = 7\pi/4$.

Hence, the general value of θ is $2n\pi + 7\pi/4, n \in \mathbb{Z}$.

3. (1) $x^4 - 2x^2 \sin^2 \frac{\pi x}{2} + 1 = 0$

$$\Rightarrow x^4 + 1 = 2x^2 \sin^2 \frac{\pi x}{2}$$

$$\Rightarrow x^4 + 1 = 2x^2 \left(1 - \cos^2 \frac{\pi x}{2} \right)$$

$$\Rightarrow (x^2 - 1)^2 + 2x^2 \cos^2 \frac{\pi x}{2} = 0$$

$$\Rightarrow (x^2 - 1) = 0 \text{ and } 2x^2 \cos^2 \frac{\pi x}{2} = 0$$

$$\Rightarrow x = \pm 1$$

4. (3) $2\sin^2 \theta - \cos 2\theta = 0$

$$\Rightarrow 2\sin^2 \theta - 1 + 2\sin^2 \theta = 0$$

$$\Rightarrow \sin \theta = \pm \frac{1}{2}$$

$$2\cos^2 \theta - 3\sin \theta = 0$$

$$\Rightarrow 2\sin^2 \theta + 3\sin \theta - 2 = 0$$

$$\Rightarrow \sin \theta = \frac{1}{2} \text{ or } -2$$

$$\therefore \sin \theta = \frac{1}{2} \text{ (common values)}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$(\therefore \theta \in [0, 2\pi])$$

5. (3) $(\cos^2 x - \sin^2 x) \left(2\sin \frac{x}{2} - 1 \right) = 0$

$$\cos 2x = 0 \text{ or } \sin \frac{x}{2} = \frac{1}{2}$$

$$\therefore \text{Number of solutions} = 8 + 2 = 10$$

6. (3) $(4 \cos^2 2x + 4 \cos 2x + 1) + (\tan^2 x - 2\sqrt{3} \tan x + 3) = 0$

$$\Rightarrow (2 \cos 2x + 1)^2 + (\tan x - \sqrt{3})^2 = 0$$

$$\Rightarrow \cos 2x = -\frac{1}{2} \text{ and } \tan x = \sqrt{3}$$

$$\Rightarrow 2\cos^2 x - 1 = -\frac{1}{2} \text{ and } \tan x = \sqrt{3}$$

$$\Rightarrow \cos x = \pm \frac{1}{2} \text{ and } \tan x = \sqrt{3}$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3}$$

$$\therefore \text{Number of solutions of equation} = 2$$

7. (1) $3 + 3 \cos \theta = 2 - 2 \cos^2 \theta$

$$\Rightarrow 2\cos^2 \theta + 3 \cos \theta + 1 = 0$$

$$\Rightarrow (2 \cos \theta + 1)(\cos \theta + 1) = 0$$

$$\begin{aligned} \therefore \cos \theta &= -1 \text{ or } \cos \theta = -\frac{1}{2} \\ \text{If } \cos \theta &= -1 \text{ then } \theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi, \dots \\ \text{If } \cos \theta &= -1/2 \text{ then } \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \dots \end{aligned}$$

Solutions in increasing order are

$$\frac{2\pi}{3} < \pi < \frac{4\pi}{3} < \frac{8\pi}{3} < 3\pi < \frac{10\pi}{3} < \frac{14\pi}{3} < 5\pi, \dots$$

$$\therefore \theta_3 + \theta_7 = \frac{4\pi}{3} + \frac{14\pi}{3} = 6\pi$$

$$8. (3) \theta = k\pi, k = \frac{p}{q}, p, q \in I, q \neq 0$$

$\cos k\pi$ is a rational. Hence,
 $k = 0, 1, 1/2, 1/3, 2/3$

There are five values of $\cos \theta$ for which $\cos \theta$ is rational.

$$9. (1) x \in [0, 2\pi] \text{ and } y \in [0, 2\pi], \text{ and } \sin x + \sin y = 2$$

This is possible only when $\sin x = 1$ and $\sin y = 1$. Thus,
 $x = \pi/2$ and $y = \pi/2$

Hence, $x + y = \pi$.

$$10. (2) 1 - \sin^2 x + \frac{\sqrt{3}+1}{2} \sin x - \frac{\sqrt{3}}{4} - 1 = 0$$

$$\text{or } \sin^2 x - \frac{\sqrt{3}+1}{2} \sin x + \frac{\sqrt{3}}{4} = 0$$

$$\text{or } 4 \sin^2 x - 2\sqrt{3} \sin x - 2 \sin x + \sqrt{3} = 0$$

On solving, we get $\sin x = 1/2, \sqrt{3}/2$

$$\Rightarrow x = \pi/6, 5\pi/6; \pi/3, 2\pi/3$$

$$11. (1) \text{ From the given relation}$$

$$\cos \theta = (2 \sin \theta \cos \theta) \sin \theta, \sin \theta \neq 0$$

$$\Rightarrow \sin \theta = \pm \frac{1}{\sqrt{2}} \text{ or } \cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}$$

($\because \theta \in [0, \pi]$)

Then the sum of roots is $3\pi/2$.

$$12. (3) \text{ The given equation is}$$

$$(\cos x - 1)(12\cos^2 x + 5\cos x + 9) = 0$$

$\Rightarrow \cos x = 1$ only as the other factor gives imaginary roots

$$\Rightarrow x = 2n\pi, n \in Z$$

Hence, it has infinite solutions as $n \in Z$.

$$13. (4) \sin x + \cos x - 2\sqrt{2} \sin x \cos x = 0$$

$$\Rightarrow \sin x + \cos x = 2\sqrt{2} \sin x \cos x$$

$$\Rightarrow 1 + \sin 2x = 2\sin^2 2x$$

$$\Rightarrow 2\sin^2 2x - \sin 2x - 1 = 0$$

$$\Rightarrow (2\sin 2x + 1)(\sin 2x - 1) = 0$$

$$\Rightarrow \sin 2x = -1/2 \text{ or } \sin 2x = 1$$

$$\Rightarrow 2x = 11\pi/6, \pi/2$$

$$\Rightarrow x = 11\pi/12, \pi/4$$

$$14. (1) \frac{\tan 5x - \tan 4x}{1 + \tan 5x \tan 4x} = 1$$

$$\Rightarrow \tan(5x - 4x) = 1$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$$

$$15. (1) a, b, c \text{ are roots of equation.}$$

$$x \sin \theta + y \sin 2\theta + z \sin 3\theta = \sin 4\theta$$

$$\Rightarrow x \sin \theta + y(2 \sin \theta \cos \theta) + z(3 \sin \theta - 4 \sin^3 \theta) = 4 \sin \theta \cos \theta \cos 2\theta$$

$$\Rightarrow \cos^3 \theta - \frac{z}{2} \cos^2 \theta - \frac{y+2}{4} \cos \theta + \frac{z-x}{8} = 0$$

$$16. (1) \sin 2\theta - 2 \cos \theta + 4 \sin \theta = 4$$

$$\Rightarrow (2 \sin \theta \cos \theta + 4 \sin \theta) = (2 \cos \theta + 4)$$

$$\Rightarrow \sin \theta (\cos \theta + 2) = (\cos \theta + 2)$$

$$\therefore \sin \theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$17. (2) \text{ We have } \tan \frac{2\pi x}{x^2 + x + 1} = -\sqrt{3}, x \neq 0$$

$$\Rightarrow \frac{2\pi x}{x^2 + x + 1} = n\pi - \frac{\pi}{3}, n \in Z$$

$$\Rightarrow x + \frac{1}{x} = \frac{7-3n}{3n-1}$$

$$\Rightarrow n = 0, \pm 1 \text{ (as L.H.S. } \leq -2 \text{ or } \geq 2)$$

$$18. (2) \text{ We have } \log_{\cos x} \left(\frac{\sqrt{3}}{2} \sin x \right) - \log_{\cos x} (\tan x) = 2$$

$$\Rightarrow \log_{\cos x} \left(\frac{\sqrt{3}}{2} \cos x \right) = 2$$

$$\Rightarrow \frac{\sqrt{3}}{2} \cos x = \cos^2 x$$

$$\Rightarrow \cos x \left(\cos x - \frac{\sqrt{3}}{2} \right) = 0$$

$$\therefore x = \frac{\pi}{6} \quad \left(x = \frac{\pi}{2} \text{ is rejected} \right)$$

$$19. (2) \sin^4 x - \cos^2 x \sin x + 2 \sin^2 x + \sin x = 0$$

$$\text{or } \sin x [\sin^3 x - \cos^2 x + 2 \sin x + 1] = 0$$

$$\text{or } \sin x [\sin^3 x - 1 + \sin^2 x + 2 \sin x + 1] = 0$$

$$\text{or } \sin x [\sin^3 x + \sin^2 x + 2 \sin x] = 0$$

$$\text{or } \sin^2 x = 0 \text{ or } \sin^2 x + \sin x + 2 = 0$$

(not possible for real x)

$$\text{or } \sin x = 0$$

Hence, the solutions are $x = 0, \pi, 2\pi, 3\pi$.

$$20. (2) y = \sin x - \cos x = \sqrt{2} \left[\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x \right]$$

$$= \sqrt{2} \sin \left(x - \frac{\pi}{4} \right)$$

$$\Rightarrow -\sqrt{2} \leq y \leq \sqrt{2}$$

$$\Rightarrow \text{Range of } y \text{ is } [-\sqrt{2}, \sqrt{2}]$$

$$21. (1) \sin(\sqrt{1 + \sin 2\theta}) = \sin \theta + \cos \theta$$

$$\Rightarrow \sin(\sin \theta + \cos \theta) = \sin \theta + \cos \theta$$

The equation of the form $\sin x = x$

$$\Rightarrow x = 0$$

$$\Rightarrow 1 + \sin 2\theta = 0$$

$$\Rightarrow \sin 2\theta = -1$$

$$\Rightarrow \theta = n\pi - \frac{\pi}{4}, n \in Z$$

22. (3) We have $4 \sin \theta \sin 2\theta \sin 4\theta = 3 \sin \theta - 4 \sin^3 \theta$
 or $\sin \theta [4 \sin 2\theta \sin 4\theta - 3 + 4 \sin^2 \theta] = 0$
 or $\sin \theta [2 (\cos 2\theta - \cos 6\theta) - 3 + 2 (1 - \cos 2\theta)] = 0$
 or $\sin \theta (-2 \cos 6\theta - 1) = 0$
 or $\sin \theta = 0$ or $\cos 6\theta = -1/2$
 $\Rightarrow \theta = n\pi$ or $6\theta = 2n\pi \pm 2\pi/3, \forall n \in \mathbb{Z}$
 $= n\pi$ or $\theta = (3n \pm 1)\pi/9, \forall n \in \mathbb{Z}$

23. (3) The given equation is
 $8 \sin x \cos x \cos 2x \cos 4x = \sin 6x$ ($\sin x \neq 0$)
 $\Rightarrow \sin 8x = \sin 6x$
 $\Rightarrow 2 \cos 7x \sin x = 0$
 As $\sin x \neq 0, \cos 7x = 0$
 or $7x = n\pi + \pi/2, n \in \mathbb{Z}$
 i.e., $x = n\pi/7 + \pi/14, n \in \mathbb{Z}$

24. (2) $(\sin^2 \theta + \cos^2 \theta + \sin \theta \cos \theta) - \frac{\cos \theta}{|\operatorname{cosec} \theta|} - 2 = -1$
 or $1 + \sin \theta \cos \theta - |\sin \theta| \cos \theta - 1 = 0$
 or $\cos \theta (\sin \theta - |\sin \theta|) = 0$
 $\Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$ or $\theta \in \left(\frac{\pi}{2}, \pi\right)$
 But $\theta \neq \frac{\pi}{4}$ (not in domain); therefore, $\theta \in \left(\frac{\pi}{2}, \pi\right)$

25 (3) We have $\cos y = \frac{\sqrt{3}}{2}$

$$\Rightarrow y = \frac{\pi}{6}$$

Let $\cot(x - y) = t$

$$\therefore t^2 - t - \sqrt{3}t + \sqrt{3} = 0$$

$$\Rightarrow (t - 1)(t - \sqrt{3}) = 0$$

$$\therefore \cot(x - y) = 1 \text{ or } \cot(x - y) = \sqrt{3}$$

Given that $x \in (0, \pi)$

If $x - y = \frac{\pi}{6}$ then $x = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3} \Rightarrow (x, y) = \left(\frac{\pi}{3}, \frac{\pi}{6}\right)$

If $x - y = \frac{\pi}{4}$ then $x = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12} \Rightarrow (x, y) = \left(\frac{5\pi}{12}, \frac{\pi}{6}\right)$

Possible ordered pairs (x, y) are $\left(\frac{\pi}{3}, \frac{\pi}{6}\right), \left(\frac{5\pi}{12}, \frac{\pi}{6}\right)$.

26. (1) $\cot\left(\frac{\pi}{3\sqrt{3}} \sin 2x\right) = \sqrt{3}$

$$\Rightarrow \frac{\pi}{3\sqrt{3}} \sin 2x = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$$

$$\Rightarrow \sin 2x = 3\sqrt{3}n + \frac{\sqrt{3}}{2}$$

For solution, $n = 0$

$$\Rightarrow \sin 2x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow 2x = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{6}$$

27. (3) The given equation becomes
 $\sin \theta + \cos \theta = \sqrt{15} \sin \theta \cos \theta$
 Squaring, $1 + \sin 2\theta = 15 \sin^2 \theta \cos^2 \theta$

$$1 + \sin 2\theta = \frac{15}{4} \sin^2 2\theta$$

$$\Rightarrow 15 \sin^2 2\theta - 4 \sin 2\theta - 4 = 0$$

$$\Rightarrow \sin 2\theta = \frac{2}{3}, -\frac{2}{5}$$

As θ varies from 0 to π , 2θ varies from 0 to 2π .

Corresponding to $\sin 2\theta = \frac{2}{3}$, there are two solutions.

Corresponding to $\sin 2\theta = -\frac{2}{5}$, there are two solutions.

28. (1) The given equation is

$$3(\sin x + \cos x) - 2(\sin x + \cos x)(1 - \sin x \cos x) = 8$$

$$\text{or } (\sin x + \cos x)[3 - 2 + 2 \sin x \cos x] = 8$$

$$\text{or } (\sin x + \cos x)[\sin^2 x + \cos^2 x + 2 \sin x \cos x] = 8$$

$$\text{or } (\sin x + \cos x)^3 = 8$$

$$\text{or } \sin x + \cos x = 2$$

or $\sin x, \cos x = 1$, which is not possible. Hence, the given equation has no solution.

29. (2) The given equation is equivalent to

$$2 \sin^2((\pi/2) \cos^2 x) = 2 \sin^2((\pi/2) \sin 2x)$$

$$\text{or } \cos^2 x = \sin 2x$$

$$\text{or } \cos x (\cos x - 2 \sin x) = 0$$

$$\Rightarrow 1 - 2 \tan x = 0 \text{ as } \cos x \neq 0, x \neq (2n+1)\frac{\pi}{2}$$

$$\text{or } \tan x = \frac{1}{2}$$

$$\Rightarrow \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{3}{5}$$

30. (4) $\cos 6x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$

$$\Rightarrow 2 \sin 2x \cdot \sin 4x = 0$$

$$\Rightarrow \sin x \cdot \cos x \cdot \sin 4x = 0$$

$$\Rightarrow x = n\pi, 4x = m\pi, m, n \in \mathbb{Z}$$

$$\Rightarrow x = n\pi, x = \frac{m\pi}{4}; m, n \in \mathbb{Z}$$

$$\therefore x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$$

31. (1) $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$

$$\text{or } \sin x \cos x (\sin^2 x + \sin x \cos x + \cos^2 x) = 1$$

$$\text{or } \frac{\sin 2x}{2} \left(1 + \frac{\sin 2x}{2}\right) = 1$$

$$\text{or } \sin 2x (2 + \sin 2x) = 4$$

$$\text{or } \sin^2 2x + 2 \sin 2x - 4 = 0$$

$$\text{or } \sin 2x = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$$

This is not possible since $-1 \leq \sin 2x \leq 1$.

Hence, the given equation has no solution.

32. (2) $2 \cos \theta [\cos 120^\circ + \cos 2\theta] = 1$

$$\text{or } 2 \cos \theta \left(-\frac{1}{2} + 2 \cos^2 \theta - 1\right) = 1$$

$$\text{or } 4 \cos^3 \theta - 3 \cos \theta - 1 = 0$$

$$\text{or } \cos 3\theta = 1 = \cos 0$$

$$\text{or } 3\theta = 2n\pi \text{ or } \theta = \frac{2n\pi}{3}, n \in \mathbb{Z}$$

Given the values so that $2n$ does not exceed 18.

$$\therefore n = 0, 1, 2, 3, \dots, 9$$

$$\text{Hence, the sum} = \frac{2\pi}{3} \sum_{n=1}^9 n = \frac{2\pi}{3} \times \frac{9(9+1)}{2} = 30\pi.$$

(2) Dividing the given equation by $\cos^2 x$, as $\cos x = 0$ does not satisfy the equation, we have

$$\tan^2 x - 5 \tan x - 6 = 0$$

$$\text{or } (\tan x + 1)(\tan x - 6) = 0$$

$$\text{or } \tan x = -1 \text{ or } \tan x = 6$$

$$\text{or } \tan x = -1 = \tan(-\pi/4), \text{ then}$$

$$x = n\pi - \pi/4, \forall n \in \mathbb{Z}$$

$$\text{and, if } \tan x = 6 = \tan \alpha \text{ (say)}$$

$$\Rightarrow \alpha = \tan^{-1} 6, \text{ then}$$

$$x = n\pi + \alpha = n\pi + \tan^{-1} 6, \forall n \in \mathbb{Z}$$

$$\text{Hence, } x = n\pi - (\pi/4), n\pi + \tan^{-1} 6, n \in \mathbb{Z}.$$

$$(1) \sin^4 x + \cos^4 x = \sin x \cos x$$

$$\text{or } (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = \sin x \cos x$$

$$\text{or } 1 - \frac{\sin^2 2x}{2} = \frac{\sin 2x}{2}$$

$$\text{or } \sin^2 2x + \sin 2x - 2 = 0$$

$$\text{or } (\sin 2x + 2)(\sin 2x - 1) = 0$$

$$\text{or } \sin 2x = 1$$

$$\text{or } 2x = (4n+1) \frac{\pi}{2}, n \in \mathbb{Z}$$

$$\text{or } x = (4n+1) \frac{\pi}{4}, n \in \mathbb{Z}$$

$$= \frac{\pi}{4}, \frac{5\pi}{4} \quad (\because x \in [0, 2\pi])$$

Thus, there are two solutions.

(4) From the given equation, we have

$$\frac{\tan \theta + \tan 4\theta}{1 - \tan \theta \tan 4\theta} = -\tan 7\theta$$

$$\text{or } \tan(\theta + 4\theta) = -\tan 7\theta$$

$$\text{or } \tan 5\theta = \tan(-7\theta)$$

$$\Rightarrow 5\theta = n\pi - 7\theta$$

$$\text{or } \theta = n\pi/12, \text{ where } n \in \mathbb{Z}, \text{ but } n \neq 6, 18, 30, \dots$$

(2) From the given equation, we have

$$\tan \theta + \tan 2\theta + \tan(\theta + 2\theta) = 0$$

$$\Rightarrow (\tan \theta + \tan 2\theta) + \frac{\tan \theta + \tan 2\theta}{1 - \tan \theta \tan 2\theta} = 0$$

$$\Rightarrow (\tan \theta + \tan 2\theta) \left[1 + \frac{1}{1 - \tan \theta \tan 2\theta} \right] = 0$$

$$\Rightarrow (\tan \theta + \tan 2\theta)(2 - \tan \theta \tan 2\theta) = 0$$

$$\Rightarrow \tan \theta = \tan(-2\theta)$$

$$\text{or } 2 - \tan \theta [(2 \tan \theta)/(1 - \tan^2 \theta)] = 0$$

$$\Rightarrow \theta = n\pi - 2\theta \text{ or } 1 - 2 \tan^2 \theta = 0$$

$$\Rightarrow \theta = n\pi/3 \text{ or } \tan^2 \theta = 1/2 = \tan^2 \alpha \text{ (say)}$$

$$\text{Therefore, } \theta = n\pi \pm \alpha, \text{ where } \tan \alpha = 1/\sqrt{2}.$$

$$(\because \text{for } \theta = n\pi/6, \tan 3\theta \text{ is not defined})$$

(4) We have

$$\frac{1}{\sin^2 \theta \cos^2 \theta} + \frac{2}{\sin^2 \theta} = 8, \sin \theta \neq 0, \cos \theta \neq 0$$

$$\text{or } 1 + 2 \cos^2 \theta = 8 \sin^2 \theta \cos^2 \theta$$

$$\text{or } 1 + 2 \cos^2 \theta = 8 \cos^2 \theta (1 - \cos^2 \theta)$$

$$\text{or } 8 \cos^4 \theta - 6 \cos^2 \theta + 1 = 0$$

$$\text{or } (4 \cos^2 \theta - 1)(2 \cos^2 \theta - 1) = 0$$

$$\text{or } \cos^2 \theta = 1/4 = \cos^2(\pi/3)$$

$$\text{or } \cos^2 \theta = 1/2 = \cos^2(\pi/4)$$

$$\text{or } \theta = n\pi \pm (\pi/3) \text{ or } \theta = n\pi \pm (\pi/4), n \in \mathbb{Z}$$

$$\text{Hence, for } 0 \leq \theta \leq \pi/2, \theta = \pi/3, \theta = \pi/4$$

$$\begin{aligned} 38. (3) \text{ Let } Z &= \frac{3+2i \sin \theta}{1-2i \sin \theta} = \frac{(3+2i \sin \theta)(1+2i \sin \theta)}{(1-2i \sin \theta)(1+2i \sin \theta)} \\ &= \frac{(3-4 \sin^2 \theta) + 8i \sin \theta}{1+4 \sin^2 \theta} \end{aligned}$$

$$\text{Therefore, the real part of } Z = \frac{3-4 \sin^2 \theta}{1+4 \sin^2 \theta}$$

$$\text{and the imaginary part of } Z = \frac{8 \sin \theta}{1+4 \sin^2 \theta}$$

$$Z \text{ is real, if imaginary part} = \frac{8 \sin \theta}{1+4 \sin^2 \theta} = 0$$

$$\text{or } \sin \theta = 0 \text{ or } \theta = n\pi, \forall n \in \mathbb{I}$$

Z is purely imaginary, if real part

$$(3-4 \sin^2 \theta)/(1+4 \sin^2 \theta) = 0$$

$$\text{or } \sin^2 \theta = 3/4 = \sin^2(\pi/3)$$

$$\text{or } \theta = n\pi \pm \pi/3, \forall n \in \mathbb{I}$$

39. (4) We have

$$(\sin x + \sin 3x) + \sin 2x = (\cos x + \cos 3x) + \cos 2x$$

$$\text{or } 2 \sin 2x \cos x + \sin 2x = 2 \cos 2x \cos x + \cos 2x$$

$$\text{or } \sin 2x (2 \cos x + 1) = \cos 2x (2 \cos x + 1)$$

$$\text{or } (2 \cos x + 1)(\sin 2x - \cos 2x) = 0$$

$$\text{or } \cos x = -1/2 \text{ or } \sin 2x - \cos 2x = 0$$

$$\Rightarrow x = 2n\pi \pm (2\pi/3) \text{ or } \tan 2x = 1 = \tan(\pi/4)$$

$$\Rightarrow x = 2n\pi \pm (2\pi/3) \text{ or } x = (4n+1)\pi/8, n \in \mathbb{Z}$$

But here $0 \leq x \leq 2\pi$. Hence, $x = \pi/8, 5\pi/8, 2\pi/3, 9\pi/8, 4\pi/3, 13\pi/8$.

$$40. (1) \text{ Let } \sin 2x = y, \text{ then } 1 + y^4 = 17(1 + y)^4$$

$$\Rightarrow y = \sin 2x = -1/2$$

$$\Rightarrow x = 105^\circ, 165^\circ, 285^\circ, 345^\circ$$

$$41. (1) \tan^4 \theta + 2 \tan^2 \theta = (\tan^2 \theta + 1)^2 - 1 = (\sec^2 \theta)^2 - 1 = \sec^4 \theta - 1$$

Putting $\sec^2 \theta = t$, we get

$$(\sqrt{3})^t = t^2 - 1$$

$$\Rightarrow t = 2 \text{ is the only solution as } t > 1$$

Hence, there will be 2 values of θ in given interval.

$$42. (3) 2 \cos x + \cos 2kx = 3$$

$$\Rightarrow 2 \left(1 - 2 \sin^2 \frac{x}{2} \right) + 1 - 2 \sin^2 kx = 3$$

$$\Rightarrow 2 \sin^2 \frac{x}{2} + \sin^2 kx = 0 \quad \dots(1)$$

Since $\sin^2 \frac{x}{2}$ and $\sin^2 kx$ are both non-negative, Eq. (1) is

possible only if $\sin^2 \frac{x}{2} = 0$ and $\sin^2 kx = 0$.

$$x = 0, \pm 2\pi, \pm 4\pi, \dots$$

But for Eq. (1) to have unique solution, the possible value of k is irrational.

$\therefore k = \sqrt{2}$ is the possible option.

$$\begin{aligned}
 43. (3) \quad & \frac{1}{\sin x} - \frac{1}{\sin 2x} = \frac{2}{\sin 4x} \\
 \Rightarrow & \frac{\sin 2x - \sin x}{\sin x \sin 2x} = \frac{2}{2 \sin 2x \cos 2x} \\
 \Rightarrow & 2 \sin 2x \cos 2x - 2 \sin x \cos 2x = 2 \sin x \\
 \Rightarrow & \sin 4x - \sin 3x + \sin x = 2 \sin x \\
 \Rightarrow & \sin 4x = \sin 3x + \sin x \\
 \Rightarrow & 2 \sin 2x \cos 2x = 2 \sin 2x \cos x \\
 \Rightarrow & 2x = 2n\pi \pm x \quad (\text{as } \sin 2x \neq 0) \\
 \Rightarrow & x = \frac{2n\pi}{3}, n \in I \quad (\text{as } x = 2k\pi \text{ is not in domain}) \\
 & n = 1, 2, 4, 5, \dots
 \end{aligned}$$

Thus, four solutions are $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$.

$$\begin{aligned}
 44. (3) \quad & (1 - \tan \theta) [1 + 2 \tan \theta / (1 + \tan^2 \theta)] = 1 + \tan \theta \\
 \text{or} \quad & (1 - \tan \theta) (1 + \tan \theta)^2 = (1 + \tan \theta) (1 + \tan^2 \theta) \\
 \text{or} \quad & (1 + \tan \theta) [(1 - \tan^2 \theta) - (1 + \tan^2 \theta)] = 0 \\
 \text{or} \quad & -2 \tan^2 \theta = 0, (1 + \tan \theta) = 0 \\
 \text{or} \quad & \tan \theta = 0, \text{ or } \tan \theta = -1 \\
 \Rightarrow & \theta = n\pi \text{ or } n\pi - \pi/4, \forall n \in Z, \\
 & \text{For } \theta \in [0, 2\pi], \theta = 0, \pi, 2\pi, 3\pi/4, 7\pi/4
 \end{aligned}$$

$$\begin{aligned}
 45. (1) \quad & \tan(A - B) = 1 \\
 \Rightarrow & A - B = n_1\pi + \frac{\pi}{4} \text{ or } A - B = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, \dots \\
 & \sec(A + B) = \frac{2}{\sqrt{3}}
 \end{aligned}$$

$$\Rightarrow A + B = 2n_2\pi \pm \frac{\pi}{6} \text{ or } A + B = \frac{\pi}{6}, \frac{11\pi}{6}, \dots$$

For the least positive values of A and B , we have

$$A + B = \frac{11\pi}{6}, A - B = \frac{\pi}{4}$$

$$\Rightarrow B = \frac{19\pi}{24}, A = \frac{25\pi}{24}$$

$$\begin{aligned}
 46. (1) \quad & \text{Let } A = \theta + 15^\circ, B = \theta - 15^\circ \\
 \Rightarrow & A + B = 2\theta \text{ and } A - B = 30^\circ
 \end{aligned}$$

$$\text{Now } \frac{\tan A}{\tan B} = \frac{3}{1}$$

$$\text{or } \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{3 + 1}{3 - 1}$$

(applying componendo and dividendo rule)

$$\text{or } \frac{\sin(A + B)}{\sin(A - B)} = 2$$

$$\Rightarrow \sin 2\theta = 2 \sin 30^\circ = 1$$

$$\Rightarrow 2\theta = 2n\pi + \frac{\pi}{2} \text{ or } \theta = n\pi + \frac{\pi}{4}, n \in Z$$

$$47. (1) \quad \tan 3\theta + \tan \theta = 2 \tan 2\theta$$

$$\text{or } \tan 3\theta - \tan 2\theta = \tan 2\theta - \tan \theta$$

$$\text{or } \frac{\sin(3\theta - 2\theta)}{\cos 3\theta \cos 2\theta} = \frac{\sin(2\theta - \theta)}{\cos 2\theta \cos \theta}$$

$$\text{or } \sin \theta (2 \sin \theta \sin 2\theta) = 0$$

$$\text{or } \sin \theta = 0 \text{ or } \sin 2\theta = 0$$

$$\Rightarrow \theta = n\pi \text{ or } 2\theta = n\pi, n \in Z$$

But $\theta = n\pi/2$ is rejected because when n is odd, $\tan \theta$ is not defined and when n is even, i.e., $2r$, then $\theta = r\pi$.

Then $\theta = n\pi, n \in I$ is the only solution.

$$48. (1) \quad (2 \sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$$

$$\Rightarrow \sin^2 x = \frac{1}{2} \text{ and } \tan^2 x = 1$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{4}, n \in Z$$

$$49. (2) \quad \text{We have } \sin 3\alpha = 4 \sin \alpha (\sin^2 x - \sin^2 \alpha)$$

$$\text{or } 3 \sin \alpha - 4 \sin^3 \alpha = 4 \sin \alpha \sin^2 x - 4 \sin^3 \alpha$$

$$\text{or } 3 \sin \alpha = 4 \sin \alpha \sin^2 x$$

$$\text{If } \sin \alpha \neq 0, \sin^2 x = 3/4 = (\sqrt{3}/2)^2 = \sin^2(\pi/3);$$

$$\text{Therefore } x = n\pi \pm \pi/3, \forall n \in Z$$

If $\sin \alpha = 0$, i.e., $\alpha = n\pi$, equation becomes an identity.

$$50. (1) \quad 4 \sin^4 x + \cos^4 x = 1$$

$$\Rightarrow (2 \sin^2 x)^2 + \frac{1}{4} (2 \cos^2 x)^2 = 1$$

$$\text{or } (1 - \cos 2x)^2 + \frac{1}{4} (1 + \cos 2x)^2 = 1$$

$$\text{or } 5 \cos^2 2x - 6 \cos 2x + 1 = 0$$

$$\text{or } (\cos 2x - 1) (5 \cos 2x - 1) = 0$$

$$\text{or } \cos 2x = 1 \text{ or } \cos 2x = \frac{1}{5}$$

$$\Rightarrow 2x = 2n\pi \text{ or } 2x = 2n\pi \pm \alpha,$$

$$\text{where } \alpha = \cos^{-1}\left(\frac{1}{5}\right), \forall n \in Z$$

$$51. (1) \quad (\sqrt{3} - 1) \sin \theta + (\sqrt{3} + 1) \cos \theta = 2$$

$$\text{or } \frac{(\sqrt{3} - 1)}{2\sqrt{2}} \sin \theta + \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) \cos \theta = \frac{1}{\sqrt{2}}$$

$$\text{or } \sin \frac{\pi}{12} \sin \theta + \cos \frac{\pi}{12} \cos \theta = \cos \frac{\pi}{4}$$

$$\text{or } \cos\left(\theta - \frac{\pi}{12}\right) = \cos \frac{\pi}{4}$$

$$\Rightarrow \theta - \frac{\pi}{12} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$\text{or } \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}, n \in Z$$

$$52. (4) \quad \text{The given expression is}$$

$$\cos y \sin x - \sin y \cos x + \sin y \sin x + \cos x \cos y$$

$$\therefore \sin(x - y) + \cos(x - y) = 0$$

$$\Rightarrow \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin(x - y) + \frac{1}{\sqrt{2}} \cos(x - y) \right) = 0$$

$$\text{or } \sin\left(x - y + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \frac{\pi}{4} + x - y = n\pi, n \in Z$$

$$\Rightarrow x - y = n\pi - \frac{\pi}{4}, n \in Z$$

$$\Rightarrow x = n\pi - \frac{\pi}{4} + y, \text{ where } n \in Z$$

4. (2) We have $\sqrt{3} \cos \theta - 3 \sin \theta = 2 (\sin 5\theta - \sin \theta)$
 or $(\sqrt{3}/2) \cos \theta - (1/2) \sin \theta = \sin 5\theta$
 or $\cos(\theta + \pi/6) = \sin 5\theta = \cos(\pi/2 - 5\theta)$
 $\Rightarrow \theta + \pi/6 = 2n\pi \pm (\pi/2 - 5\theta), n \in \mathbb{Z}$
 $\Rightarrow \theta = (n\pi/3) + (\pi/18)$
 $\Rightarrow \theta = (-n\pi/2) + (\pi/6), \forall n \in \mathbb{Z}$

4. (3) $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$
 or $(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x + \sin 2x + \alpha = 0$
 $\Rightarrow \sin^2 2x - 2 \sin 2x - 2 - 2\alpha = 0$
 Let $\sin 2x = y$. Then the given equation becomes
 $y^2 - 2y - 2(1 + \alpha) = 0$, where $-1 \leq y \leq 1$,
 $(\because -1 \leq \sin 2x \leq 1)$

For real y , discriminant ≥ 0

$$\Rightarrow 3 + 2\alpha \geq 0 \quad \text{or } \alpha \geq -\frac{3}{2}$$

$$\text{Also } -1 \leq y \leq 1 \quad \text{or } -1 \leq 1 - \sqrt{3 + 2\alpha} \leq 1$$

$$\Rightarrow 3 + 2\alpha \leq 4 \quad \text{or } \alpha \leq \frac{1}{2}. \text{ Thus } -\frac{3}{2} \leq \alpha \leq \frac{1}{2}$$

55. (1) $\cos x = \sqrt{1 - \sin 2x} = |\sin x - \cos x|$

(a) $\sin x < \cos x$

$$\Rightarrow x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right] \quad \dots(i)$$

Then the given equation is

$$\cos x = \cos x - \sin x$$

or $\sin x = 0$

$$\Rightarrow x = 2\pi$$

[from Eq. (i)]

(b) $\sin x \geq \cos x$

$$\Rightarrow x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

$$\Rightarrow \cos x = \sin x - \cos x$$

or $\tan x = 2$

$$\Rightarrow x = \tan^{-1} 2$$

(ii)

Hence, there are two solutions.

56. (2) $|\cot x| = \cot x + \frac{1}{\sin x}$

If $\cot x > 0$

$$\Rightarrow \cot x = \cot x + \frac{1}{\sin x} = 0$$

$$\Rightarrow \frac{1}{\sin x} = 0, \text{ which is not possible.}$$

If $\cot x \leq 0$

$$\Rightarrow -\cot x = \cot x + \frac{1}{\sin x}$$

$$\Rightarrow -2 \cot x = \frac{1}{\sin x}$$

or $\cos x = -\frac{1}{2}$

$$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

57. (4) $\cos^2 x = 2 \cos x (1 - 3 \cos^2 x)$

or $6 \cos^3 x + \cos^2 x - 2 \cos x = 0$

or $\cos x = 0, \frac{1}{2}, -\frac{2}{3}$

$$\Rightarrow x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1} \left(-\frac{2}{3}\right) \quad (\because \alpha, \beta \text{ are } +ve)$$

If $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}$, then we have $|\alpha - \beta| = \frac{\pi}{6}$.

58. (3) On adding given equations, we get

$$7 (\cos \theta + \sin \theta) = 10$$

$$\text{or } \cos \theta + \sin \theta = \frac{10}{7}$$

But maximum value of $\cos \theta + \sin \theta = \sqrt{1^2 + 1^2} = \sqrt{2} < \frac{10}{7}$.

So, no θ is possible.

59. (1) $\cos x \cos 6x = -1$

or $2 \cos x \cos 6x = -2$

or $\cos 7x + \cos 5x = -2$

or $\cos 7x = -1$ and $\cos 5x = -1$

The value of x satisfying these two equations simultaneously and lying between 0 and 2π is π .

Therefore, the general solution is $x = 2n\pi + \pi, n \in \mathbb{Z}$. Thus,

$$x = (2n + 1)\pi, n \in \mathbb{Z}$$

60. (3) $\cos(\theta) \cos(\pi\theta) = 1$

$$\Rightarrow \cos(\theta) = 1 \text{ and } \cos(\pi\theta) = 1 \quad \dots(i)$$

or $\cos(\theta) = -1 \text{ and } \cos(\pi\theta) = -1 \quad \dots(ii)$

If $\cos(\theta) = 1 \Rightarrow \theta = 2m\pi$

and $\cos(\pi\theta) = 1 \Rightarrow \theta = 2k, k \in \mathbb{Z}$

which is possible only when $\theta = 0$.

Equation (ii) is not possible for any θ as for $\cos(\theta) = -1$, θ should be odd multiple of π , i.e., irrational and for $\cos(\pi\theta) = -1$, θ should be odd integer, i.e., rational.

Both the conditions cannot be satisfied.

Therefore, $\theta = 0$ is the only solution.

61. (2) $(\sin \theta + 2)(\sin \theta + 3)(\sin \theta + 4) = 6$

$$\text{L.H.S.} \geq 6 \text{ and R.H.S.} = 6$$

Therefore, equality only holds if

$$\sin \theta = -1 \text{ or } \theta = 3\pi/2, 7\pi/2$$

$$\therefore \text{Sum} = 5\pi \Rightarrow k = 5$$

62. (2) $\sum_{r=1}^5 \cos rx = 5$

$$\Rightarrow \cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 5$$

which is possible only when $\cos x = \cos 2x = \cos 3x = \cos 4x = \cos 5x = 1$ and is satisfied by $x = 0$ and $x = 2\pi$.

63. (4) $\cos 3x + \sin \left(2x - \frac{7\pi}{6}\right) = -2$

$$\therefore \cos 3x = -1 \text{ and } \sin \left(2x - \frac{7\pi}{6}\right) = -1$$

or $3x = (2n + 1)\pi$ and $2x - \frac{7\pi}{6} = -\frac{\pi}{2} + 2m\pi; m, n \in \mathbb{Z}$

or $x = \frac{2n\pi}{3} + \frac{\pi}{3}$ and $x = m\pi + \frac{\pi}{3}$

or $x = 2k\pi + \frac{\pi}{3}, k \in \mathbb{Z}$.

S.78 Trigonometry

64. (2) We have $\sin^{100} x - \cos^{100} x = 1$

or $\sin^{100} x = 1 + \cos^{100} x$

Since L.H.S. ≤ 1 and R.H.S. ≥ 1 , L.H.S. = R.H.S. = 1.

Then, $x = n\pi + \frac{\pi}{2}, n \in \mathbb{I}$

65. (3) $\tan x + \cot x + 1 = \cos\left(x + \frac{\pi}{4}\right)$

or $\tan x + \cot x = \cos\left(x + \frac{\pi}{4}\right) - 1$

Now $\tan x + \cot x \leq -2$ and $\cos\left(x + \frac{\pi}{4}\right) - 1 \geq -2$

It implies that equality holds when both are -2 . Thus,

$$\cos\left(x + \frac{\pi}{4}\right) = -1$$

$$\Rightarrow x + \frac{\pi}{4} = (2m+1)\pi, m \in \mathbb{Z}$$

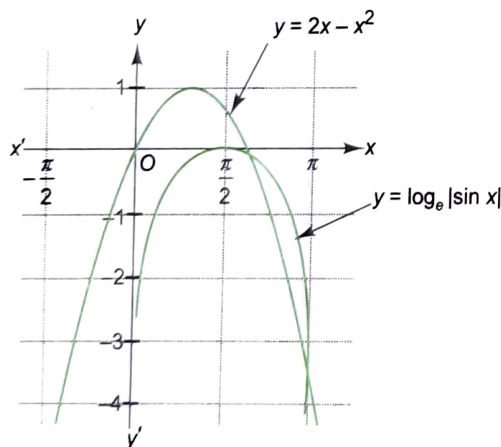
$$\Rightarrow x = \frac{3\pi}{4} \text{ or } \frac{11\pi}{4}$$

Therefore, the sum of the solutions is

$$\frac{3\pi}{4} + \frac{11\pi}{4} = \frac{7\pi}{2}$$

66. (2) $\log_e |\sin x| = -x(x-2)$

Graphs of $y = \log_e |\sin x|$ and $y = -x(x-2)$ meet exactly two times in $[0, \pi]$.

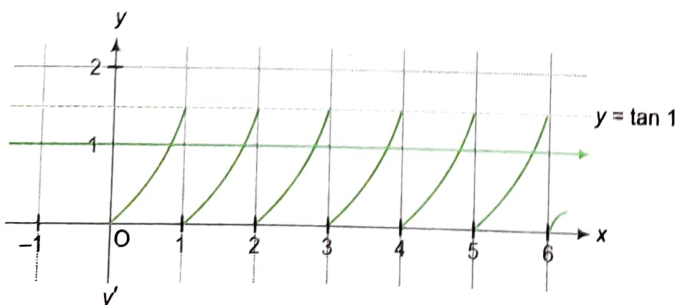


67. (2) $\sin \{x\} = \cos \{x\}$

$\tan \{x\} = 1$

$\tan(\pi/4) = 1 < \tan 1$

Graphs of $y = \tan \{x\}$ and $y = 1$ meet exactly six times in $[0, 2\pi]$.



68. (1) We have $|4 \sin x - 1| < \sqrt{5}$

$$\Rightarrow -\sqrt{5} < 4 \sin x - 1 < \sqrt{5}$$

$$\text{or } -\left(\frac{\sqrt{5}-1}{4}\right) < \sin x < \left(\frac{\sqrt{5}+1}{4}\right)$$

$$\text{or } -\sin \frac{\pi}{10} < \sin x < \cos \frac{\pi}{5}$$

$$\text{or } \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)$$

$$\text{or } \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin \frac{3\pi}{10}$$

$$\Rightarrow x \in \left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$$

69. (2) Roots will be of opposite sign if product of roots is negative

$$\Rightarrow \sin \theta < \frac{1}{2}$$

70. (1) $|2 \sin \theta - \operatorname{cosec} \theta| \geq 1$

$$\text{or } |2 \sin^2 \theta - 1| \geq |\sin \theta|$$

$$\text{or } |\cos 2\theta| \geq |\sin \theta|$$

$$\text{or } 2 \cos^2 2\theta \geq 1 - \cos 2\theta$$

$$\text{or } 2 \cos^2 2\theta + \cos 2\theta - 1 \geq 0$$

$$\text{or } (2 \cos 2\theta - 1)(\cos 2\theta + 1) \geq 0$$

$$\Rightarrow \cos 2\theta \geq \frac{1}{2} \quad [\text{as } \cos \theta \neq 0, \text{ i.e., } \cos 2\theta \neq -1]$$

71. (4) $\sin 5x = 16 \sin^5 x$

$$\text{or } \sin(3x + 2x) = 16 \sin^5 x$$

$$\text{or } 5 \sin x - 20 \sin^3 x + 16 \sin^5 x = 16 \sin^5 x$$

$$\text{or } 5 \sin x (1 - 4 \sin^2 x) = 0$$

$$\therefore \sin x = 0 \text{ or } \sin^2 x = 1/4$$

$$\therefore x = n\pi, x = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

72. (2) $|2 \sin x - \sqrt{3}|^{2 \cos^2 x - 3 \cos x + 1} = 1$

Case I: $2 \cos^2 x - 3 \cos x + 1 = 0$

$$\cos x = \frac{1}{2}, 1$$

$$x = 0, \frac{\pi}{3}$$

$$\text{But at } x = \frac{\pi}{3}; \text{ L.H.S.} = 0^0$$

$$\therefore x = \frac{\pi}{3} \text{ (rejected)}$$

$$\therefore x = 0 \text{ is a solution}$$

Case II: $2 \sin x - \sqrt{3} = 1, -1$

$$\therefore \sin x = \frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2}$$

$$\sin x = \frac{\sqrt{3}-1}{2}$$

$$\therefore x \text{ has 2 values in } [0, \pi]$$

$$\therefore \text{Total number of solutions} = 2 + 1 = 3$$

73. (1) Let $f(x) = \cos x - x + \frac{1}{2}$

$$f(0) = 1 + \frac{1}{2} > 0$$

$$f\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2} + \frac{1}{2} = \frac{1-\pi}{2} < 0$$

Therefore, one root lies in the interval $\left(0, \frac{\pi}{2}\right)$.

4. (3) Let $\log_{\cos x} \sin x = t$, then the given equation is $t + \frac{1}{t} = 2$
 or $(t-1)^2 = 0$
 or $t = 1$

or $\log_{\cos x} \sin x = 1$ or $\sin x = \cos x$
 $\Rightarrow \tan x = 1$
 or $x = \pi/4$

5. (2) Given $x^2 + 2x \sin(xy) + 1 = 0$
 or $[x + \sin(xy)]^2 + [1 - \sin^2(xy)] = 0$
 or $x + \sin(xy) = 0$ and $\sin^2(xy) = 1$
 $\sin^2(xy) = 1$ gives $\sin(xy) = 1$ or -1

If $\sin(xy) = 1 \Rightarrow x = -1$
 $\Rightarrow \sin(-y) = 1 \Rightarrow \sin y = -1$

Then the ordered pair is $(1, 3\pi/2)$.

If $\sin(xy) = -1 \Rightarrow x = 1$
 $\Rightarrow \sin y = -1 \Rightarrow (-1, 3\pi/2)$

Thus, there are two ordered pairs.

6. (1) Since the system has a non-trivial solution, the determinant of coefficients is 0. Thus,

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

or $\sin 3\theta (28 - 21) - \cos 2\theta (-7 - 7) + 2(-3 - 4) = 0$

or $\sin 3\theta + 2 \cos 2\theta - 2 = 0$

or $(3 \sin \theta - 4 \sin^3 \theta) + 2(1 - 2 \sin^2 \theta) - 2 = 0$

or $4 \sin^3 \theta + 4 \sin^2 \theta - 3 \sin \theta = 0$

or $\sin \theta (2 \sin \theta - 1) (2 \sin \theta + 3) = 0$

or $\sin \theta = 0, \frac{1}{2}$

$\therefore \theta = n\pi$ or $\theta = n\pi + (-1)^n \pi/6, \forall n \in \mathbb{Z}$

7. (2) We have $\sin^4 x - 2 \cos^2 x + a^2 = 0$

$\therefore a^2 = 2 \cos^2 x - \sin^4 x$
 $= 2 - 2 \sin^2 x - \sin^4 x$
 $= 3 - (\sin^2 x + 1)^2$

Now $0 \leq \sin^2 x \leq 1$

$\therefore 1 \leq \sin^2 x + 1 \leq 2$

$\therefore 1 \leq (\sin^2 x + 1)^2 \leq 4$

$\therefore -4 \leq -(\sin^2 x + 1)^2 \leq -1$

$\therefore -1 \leq 3 - (\sin^2 x + 1)^2 \leq 2$

$\therefore a^2 \leq 2$

$\therefore -\sqrt{2} \leq a \leq \sqrt{2}$

78. (2) $\sin^2 x + a \cos x + a^2 > 1 + \cos x$

Putting $x = 0$, we get

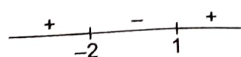
or $a + a^2 > 2$

or $a^2 + a - 2 > 0$

or $(a+2)(a-1) > 0$

or $a < -2$ or $a > 1$

Therefore, we have the largest negative integral value of $a = -3$.



79. (4) $y^2 - y + a = \left(y - \frac{1}{2}\right)^2 + a - \frac{1}{4}$

Since $-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}$, the given equation will have no real value x for any y if $a - \frac{1}{4} > \sqrt{2}$.

i.e., $a \in \left(\sqrt{2} + \frac{1}{4}, \infty\right)$

$\Rightarrow a \in (\sqrt{3}, \infty)$ (as $\sqrt{2} + \frac{1}{4} < \sqrt{3}$)

80. (3) $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$

The maximum value of the L.H.S. is 1 and the minimum of the R.H.S. is also 1. Thus,

$[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x] = 1$

or $[\sin x + \cos x] = 1, [-\sin x] = -1, [-\cos x] = -1$

$[\sin x + \cos x] = 1$

$\therefore 1 \leq \sin x + \cos x \leq \sqrt{2}$

$\therefore x \in \left[0, \frac{\pi}{2}\right]$... (i)

$[-\sin x] = -1$

$\therefore -1 \leq -\sin x < 0$

$\therefore 0 < \sin x \leq 1$... (ii)

$[-\cos x] = -1$

$\therefore -1 \leq -\cos x < 0$

$\therefore 0 < \cos x \leq 1$... (iii)

From Eqs. (i), (ii), (iii), $x \in \left(0, \frac{\pi}{2}\right)$.

81. (3) $\cos^8 x + b \cos^4 x + 1 = 0$

$\Rightarrow b = -\left(\cos^4 x + \frac{1}{\cos^4 x}\right) \leq -2 \forall x \in \mathbb{R}$

$\Rightarrow b \in (-\infty, -2]$

82. (2) We have $|\sin 2x| + |\cos 2x| = |\sin y|$

on squaring, we get $1 + |\sin 4x| = \sin^2 y$

L.H.S. ≥ 1 , but R.H.S. ≤ 1

So solution is possible only when $|\sin y| = 1$. Thus,

$\sin y = \pm 1 \Rightarrow y = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$

83. (2) Given that $|\sin x|^2 + |\sin x| + b = 0$

$\Rightarrow |\sin x| = \frac{-1 \pm \sqrt{1-4b}}{2}$

$\Rightarrow 0 \leq \frac{-1 \pm \sqrt{1-4b}}{2} < 1 \Rightarrow -2 < b < 0$

84. (4) $e^{|\sin x|} + e^{-|\sin x|} + 4a = 0$. Let $t = e^{|\sin x|}$

$\Rightarrow t \in [1, e]$

$\Rightarrow t + \frac{1}{t} + 4a = 0$

$\Rightarrow t + \frac{1}{t} = -4a$

Let, $f(t) = t + \frac{1}{t}$

$\therefore f'(t) = 1 - \frac{1}{t^2} \geq 0$ for $t \in [1, e]$

So, $f(t)$ is increasing function for $t \in [1, e]$.

$$\therefore f(t) \in [f(1), f(e)]$$

$$\text{or } f(t) \in \left[2, \frac{e^2+1}{e}\right]$$

$$\therefore -4a \in \left[2, \frac{e^2+1}{e}\right]$$

$$\therefore a \in \left[\frac{-(e^2+1)}{4e}, \frac{-1}{2}\right]$$

$$85. (3) \tan^4 x - 2 \sec^2 x + a = 0$$

$$\text{or } \tan^4 x - 2(1 + \tan^2 x) + a = 0$$

$$\text{or } \tan^4 x - 2 \tan^2 x + 1 = 3 - a$$

$$\text{or } (\tan^2 x - 1)^2 = 3 - a$$

$$\Rightarrow 3 - a \geq 0 \quad \text{or} \quad a \leq 3$$

$$86. (3) |x| + |y| = 2, \sin\left(\frac{\pi x^2}{3}\right) = 1$$

$$\Rightarrow |x|, |y| \in [0, 2], \frac{\pi x^2}{3} = (4n+1) \frac{\pi}{2}, n \in \mathbb{Z}$$

$$\Rightarrow x^2 = \frac{3(4n+1)}{2} = \frac{3}{2}, \text{ as } |x| \leq 2$$

$$\Rightarrow |x| = \sqrt{\frac{3}{2}}, |y| = 4 - \sqrt{\frac{3}{2}}$$

Thus, there are four ordered pairs.

$$87. (1) x^2 + 4 - 2x + 3 \sin(ax+b) = 0$$

$$(x-1)^2 + 3 = -3 \sin(ax+b)$$

$$\text{L.H.S.} \geq 3 \text{ and R.H.S.} \leq 3$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.} = 3$$

$$(x-1)^2 + 3 + 3 \sin(ax+b) = 0$$

$$\Rightarrow x = 1, \sin(ax+b) = -1$$

$$\Rightarrow \sin(a+b) = -1$$

$$\Rightarrow a+b = (4n-1) \frac{\pi}{2}, n \in \mathbb{I}$$

$$\Rightarrow a+b = \frac{7\pi}{2} \quad (\text{from the given options})$$

$$88. (3) \pi \log_3 \left(\frac{1}{x}\right) = k\pi, k \in \mathbb{I}$$

$$\log_3 \left(\frac{1}{x}\right) = k \text{ or } x = 3^{-k}$$

The possible values of k are $-1, 0, 1, 2, 3, \dots$

$$S = 3 + 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots = \frac{3}{1-\frac{1}{3}} = \frac{9}{2}$$

$$89. (1) \frac{\sin(xy)}{\cos(xy)} = y$$

$$\text{or } \sin(xy) = xy$$

$$\text{or } xy = 0$$

$$\therefore x = 0 \text{ or } y = 0$$

But $x = 0$ is not possible

$$\therefore y = 0 \text{ and } x = 1, \text{ i.e., } (1, 0).$$

$$90. (1) \text{ If solution lies on the line } y = 3x, \text{ then } 3 \sin 3x + 12 \sin^3 x = a$$

$$\Rightarrow 3(3 \sin x - 4 \sin^3 x) + 12 \sin^3 x = a$$

$$\Rightarrow \sin x = \frac{a}{9}$$

Hence, for no solution, $a > 9$ or $a < -9$.

Multiple Correct Answers Type

1. (1), (2)

$$\text{We have } 4 \sin^4 x + \cos^4 x = 1$$

$$\text{or } 4 \sin^4 x = 1 - \cos^4 x \\ = (1 - \cos^2 x)(1 + \cos^2 x) \\ = \sin^2 x (2 - \sin^2 x)$$

$$\text{or } \sin^2 x [5 \sin^2 x - 2] = 0$$

$$\text{or } \sin x = 0 \text{ or } \sin x = \pm \sqrt{2/5}$$

$$\Rightarrow x = n\pi \text{ or } x = n\pi \pm \sin^{-1} \sqrt{\frac{2}{5}}, n \in \mathbb{Z}$$

2. (1), (2)

$$(\sin^3 \theta + \cos^3 \theta) - (1 - \sin \theta \cos \theta) = 0$$

$$\text{or } (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta) - (1 - \sin \theta \cos \theta) = 0$$

$$\text{or } (1 - \sin \theta \cos \theta)(\sin \theta + \cos \theta - 1) = 0$$

$$\text{If } \sin \theta \cos \theta = 1$$

$$\Rightarrow 2 \sin \theta \cos \theta = 2$$

$$\Rightarrow \sin 2\theta = 2 \text{ (not possible)}$$

$$\Rightarrow \sin \theta + \cos \theta = 1$$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\Rightarrow \theta = 2n\pi \text{ or } 2n\pi + \frac{\pi}{2}$$

3. (1), (3), (4)

We have

$$\tan^2 \theta = 1 - \cos 2\theta = 2 \sin^2 \theta \text{ or } \operatorname{cosec}^2 \theta \tan^2 \theta = 2$$

$$\text{or } (1 + \cot^2 \theta) \tan^2 \theta = 2 \text{ or } \tan^2 \theta + 1 = 2$$

$$\text{or } \tan^2 \theta = 1$$

$$\text{or } \tan \theta = \pm 1$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\text{Moreover, } \tan^2 \theta = 2 \sin^2 \theta$$

$$\Rightarrow \sin^2 \theta = 0 \Rightarrow \theta = n\pi$$

4. (1), (3)

$$y + \frac{1}{y} \geq 2$$

$$\Rightarrow \sqrt{y + \frac{1}{y}} \geq \sqrt{2}$$

$$\text{But } \sin x + \cos x \leq \sqrt{2}$$

$$\Rightarrow y + \frac{1}{y} = 2 \text{ and } \sin x + \cos x = \sqrt{2}$$

$$\Rightarrow y = 1 \text{ and } \sin\left(x + \frac{\pi}{4}\right) = 1$$

$$\Rightarrow y = 1 \text{ and } x = \frac{\pi}{4}$$

5. (1), (2), (3)

$$2\sin^2\left(\frac{\pi}{2}\cos^2 x\right) = 1 - \cos(\pi \sin 2x)$$

$$\Rightarrow 2\sin^2\left\{\left(\frac{\pi}{2}\right)\cos^2 x\right\} = 2\sin^2\left\{\left(\frac{\pi}{2}\right)\sin 2x\right\}$$

$$\Rightarrow \left(\frac{\pi}{2}\right)\cos^2 x = \pm \left(\frac{\pi}{2}\right)\sin 2x$$

$$\Rightarrow \cos x = 0 \text{ or } \cos x = \pm 2 \sin x$$

$$\Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z \text{ or } \tan x = \pm \frac{1}{2}$$

6. (1), (3)

$$\sin^2 x - 2 \sin x - 1 = 0$$

$$\Rightarrow (\sin x - 1)^2 = 2 \text{ or } \sin x - 1 = \pm \sqrt{2}$$

or $\sin x = 1 - \sqrt{2}$ as $\sin x \neq 1$

There are two solutions in $[0, 2\pi]$ and two more in $[2\pi, 4\pi]$.

Thus, $n = 4, 5$.

7. (1), (2), (3)

The given equation is

$$2(\sin x + \sin y) - 2\cos(x-y) = 3$$

$$\Rightarrow 2 \times 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} - 2 \left[2\cos^2 \frac{x-y}{2} - 1 \right] = 3$$

or $4\cos^2\left(\frac{x-y}{2}\right) - 4\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 1 = 0$

$$\text{or } \cos\left(\frac{x-y}{2}\right) = \frac{4\sin\left(\frac{x+y}{2}\right) \pm \sqrt{16\sin^2\left(\frac{x+y}{2}\right) - 16}}{8}$$

$$\therefore \sin^2\left(\frac{x+y}{2}\right) \geq 1$$

$$\Rightarrow \sin \frac{x+y}{2} = \pm 1$$

Since x and y are smallest and positive, we have

$$\sin \frac{x+y}{2} = 1 \text{ and } \frac{x+y}{2} = \frac{\pi}{2}$$

$$\text{i.e., } x+y = \pi$$

$$\text{Also, } \cos\left(\frac{x-y}{2}\right) = \frac{1}{2}$$

$$\Rightarrow x-y = \frac{2\pi}{3} \text{ or } -\frac{2\pi}{3}$$

...(i)

...(ii)

From Eqs. (i) and (ii), we get

$$(x = 5\pi/6, y = \pi/6) \text{ or } (x = \pi/6, y = 5\pi/6)$$

8. (1), (4)

$$1 - 2x - x^2 = \tan^2(x+y) + \cot^2(x+y)$$

$$\Rightarrow -(x+1)^2 = [\tan(x+y) - \cot(x+y)]^2$$

Now L.H.S. ≤ 0 and R.H.S. ≥ 0

$$\Rightarrow -(x+1)^2 = [\tan(x+y) - \cot(x+y)]^2 = 0$$

$$\Rightarrow x = -1 \text{ and } \tan(x+y) = \cot(x+y)$$

$$\Rightarrow x = -1 \text{ and } \tan^2(-1+y) = 1$$

$$\Rightarrow x = -1 \text{ and } -1+y = n\pi \pm (\pi/4), n \in Z$$

9. (2), (3)

From $\tan x + \tan y = 1$, we have

$$\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = 1$$

or $\sin x \cos y + \sin y \cos x = \cos x \cos y$

or $2 \sin(x+y) = 2 \cos x \cos y$

or $2 \sin(x+y) = \cos(x+y) + \cos(x-y)$

or $2 \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + \cos(x-y)$

or $\cos(x-y) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$

$$\Rightarrow x-y = 2n\pi \pm (\pi/4), \forall n \in Z$$

...(i)

Also we have $x+y = \pi/4$

...(ii)

From Eqs. (i) and (ii), we have

$$x = n\pi + (\pi/4) \text{ and } y = -n\pi, \forall n \in Z$$

or $x = n\pi$ and $y = -n\pi + \pi/4, \forall n \in Z$

10. (1), (3)

$$x+y = 2\pi/3 \text{ or } y = (2\pi/3) - x$$

$$\therefore \sin x = 2 \sin\left(\frac{2\pi}{3} - x\right)$$

$$= 2 \left[\left(\frac{\sqrt{3}}{2}\right) \cos x + \left(\frac{1}{2}\right) \sin x \right]$$

$$= \sqrt{3} \cos x + \sin x$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = n\pi + \frac{\pi}{2}, n \in Z$$

$$\Rightarrow y = \frac{2\pi}{3} - n\pi - \frac{\pi}{2} = \frac{\pi}{6} - n\pi$$

Hence, for $x \in [0, 4\pi]$, $x = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$
and for $y \in [0, 4\pi]$, $y = \pi/6, 7\pi/6, 13\pi/6, 19\pi/6$

11. (1), (4)

$$\tan x - \tan^2 x > 0$$

or $\tan x (\tan x - 1) < 0$

or $0 < \tan x < 1$

or $0 < x < \pi/4$

$$\Rightarrow n\pi < x < n\pi + \pi/4, n \in Z \text{ (generalizing)}$$

$$|\sin x| < \frac{1}{2} \Rightarrow -\frac{1}{2} < \sin x < \frac{1}{2}$$

$$\Rightarrow -\pi/6 < x < \pi/6$$

$$\Rightarrow -\pi/6 + n\pi < x < \pi/6 + n\pi, n \in Z \text{ (generalizing)}$$

Then the common values are $n\pi < x < n\pi + \pi/6$.

12. (1), (2), (4)

$$\cos(x + \pi/3) + \cos x = a$$

or $\frac{1}{2} \cos x - \left(\frac{\sqrt{3}}{2}\right) \sin x + \cos x = a$

or $\left(\frac{3}{2}\right) \cos x - \left(\frac{\sqrt{3}}{2}\right) \sin x = a$

$$\Rightarrow -\sqrt{\left(\frac{9}{4} + \frac{3}{4}\right)} \leq a \leq \sqrt{\left(\frac{9}{4} + \frac{3}{4}\right)}$$

$$\Rightarrow -\sqrt{3} \leq a \leq \sqrt{3}$$

...(i)

Hence, there are three integral values of $a = -1, 0, 1$ whose sum is 0.

For $a = 1$, the given equation is $(\sqrt{3}/2) \cos x - (1/2) \sin x = 1/\sqrt{3}$.

Thus,

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow x + \frac{\pi}{6} = 2n\pi \pm \alpha, \text{ where } \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \alpha$$

Hence, the solutions for $a = 1$ in $[0, 2\pi]$ are

$$\cos^{-1}(1/\sqrt{3}) - 11\pi/6, 11\pi/6 - \cos^{-1}(1/\sqrt{3}).$$

13. (1), (2), (3)

The given inequality can be written as

$$2^{\operatorname{cosec}^2 x} \sqrt{(y-1)^2 + 1} \leq 2 \quad \dots(i)$$

Since $\operatorname{cosec}^2 x \geq 1$ for all real x , we have

$$2^{\operatorname{cosec}^2 x} \geq 2 \quad \dots(ii)$$

Also $(y-1)^2 + 1 \geq 1$

$$\Rightarrow \sqrt{(y-1)^2 + 1} \geq 1 \quad \dots(iii)$$

From Eqs. (ii) and (iii), we get

$$2^{\operatorname{cosec}^2 x} \sqrt{(y-1)^2 + 1} \geq 2 \quad \dots(iv)$$

Therefore, from Eqs. (i) and (iv), equality holds only when

$$2^{\operatorname{cosec}^2 x} = 2 \text{ and } \sqrt{(y-1)^2 + 1} = 1. \text{ Thus,}$$

$$\operatorname{cosec}^2 x = 1 \text{ and } (y-1)^2 + 1 = 1$$

$$\Rightarrow \sin x = \pm 1 \text{ and } y = 1$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ and } y = 1$$

Hence, the solution of the given inequality is $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $y = 1$.

14. (1), (2), (3)

$\sin^2 x - a \sin x + b = 0$ has only one solution in $(0, \pi)$.

$\Rightarrow \sin x = 1$ gives one solution, and $\sin x = \alpha$ gives other solution such that $\alpha > 1$ or $\alpha \leq 0$

$\Rightarrow (\sin x - 1)(\sin x - \alpha) = 0$ is the same equation as $\sin^2 x - a \sin x + b = 0$

$$\Rightarrow 1 + \alpha = a \text{ and } \alpha = b$$

$$\Rightarrow 1 + b = a \text{ and } b > 1 \text{ or } b \leq 0$$

$$\Rightarrow b \in (-\infty, 0] \cup [1, \infty)$$

$$\text{and } a \in (-\infty, 1] \cup [2, \infty)$$

15. (1), (2)

Given the quadratic equation is an identity. Therefore,

$$\operatorname{cosec}^2 \theta = 4 \text{ and } \cot \theta = -\sqrt{3}$$

$$\Rightarrow \operatorname{cosec} \theta = 2 \text{ or } -2 \text{ and } \tan \theta = -\frac{1}{\sqrt{3}}$$

$$\theta = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6}$$

16. (3), (4)

Abscissa of the vertex is given by

$$x = \frac{1}{\sin \alpha} > 1$$

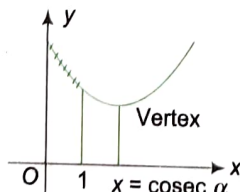
The graph of $f(x) = (\sin \alpha)x^2 - 2x + b - 2$ is shown in figure.

Therefore, the minimum of $f(x) = (\sin \alpha)x^2 - 2x + b - 2$ must be greater than zero, but minimum is at $x = 1$, i.e.,

$$\sin \alpha - 2 + b - 2 \geq 0$$

$$\Rightarrow b \geq 4 - \sin \alpha, \alpha \in (0, \pi)$$

$$\Rightarrow b \geq 4 \text{ as } \sin \alpha > 0 \text{ in } (0, \pi)$$



17. (1), (4)

$$\frac{\sqrt{3}-1}{2\sqrt{2}\sin x} + \frac{\sqrt{3}+1}{2\sqrt{2}\cos x} = 2$$

$$\sin \frac{\pi}{12} \cos x + \cos \frac{\pi}{12} \sin x = \sin 2x$$

$$\sin 2x = \sin \left(x + \frac{\pi}{12}\right)$$

$$\therefore 2x = x + \frac{\pi}{12} \text{ or } 2x = \pi - x - \frac{\pi}{12}$$

$$x = \frac{\pi}{12} \text{ or } 3x = \frac{\pi}{12}$$

$$\therefore x = \frac{\pi}{12} \text{ or } \frac{11\pi}{36}$$

18. (1), (3), (4)

$$\cos 3\theta = \cos 3\alpha$$

$$3\alpha = 2n\pi \pm 3\alpha, n \in \mathbb{Z}$$

Putting $n = 0, 1$, we have

$$3\theta = 3\alpha \text{ or } -3\alpha \text{ or } 2\pi + 3\alpha \text{ or } 2\pi - 3\alpha$$

$$\theta = \alpha \text{ or } -\alpha \text{ or } \theta = \frac{2\pi}{3} + \alpha \text{ or } \frac{2\pi}{3} - \alpha$$

Hence, options (1), (3), (4) are correct

If $n = -1$, then $3\theta = -2\pi \pm 3\alpha$

$$\Rightarrow \theta = -\frac{2\pi}{3} \pm \alpha$$

$$\begin{aligned} \Rightarrow \sin \theta &= \sin \left(-\frac{2\pi}{3} \pm \alpha\right) = -\sin \left(\frac{2\pi}{3} \pm \alpha\right) \\ &= -\sin \left(\pi - \frac{\pi}{3} \pm \alpha\right) = -\sin \left(\frac{\pi}{3} \pm \alpha\right) \end{aligned}$$

Hence, (2) is not correct.

19. (2), (3), (4)

$$1 + \cos 3x = 2 \cos 2x$$

$$\text{or } 1 + 4 \cos^3 x - 3 \cos x = 2(2 \cos^2 x - 1)$$

$$\text{or } 4 \cos^3 x - 4 \cos^2 x - 3 \cos x + 3 = 0$$

Let $\cos x = y$, we have

$$4y^3 - 4y^2 - 3y + 3 = 0$$

$$\text{or } 4y^2(y-1) - 3(y-1) = 0$$

$$\text{or } (y-1)(4y^2 - 3) = 0$$

$$\Rightarrow y = 1 \text{ or } y^2 = \frac{3}{4}$$

$$\Rightarrow \cos x = 1 \text{ or } \cos^2 x = \frac{3}{4}$$

$$\Rightarrow \cos x = 1 \text{ or } \cos^2 x = \cos^2 \pi/6$$

$$\Rightarrow x = 2n\pi \text{ or } x = n\pi \pm (\pi/6), n \in \mathbb{Z}$$

20. (1), (2), (3), (4)

$$\cos 3x + \cos 2x = \sin \frac{3x}{2} + \sin \frac{x}{2}, 0 < x < 2\pi$$

$$\text{or } 2 \cos \frac{5x}{2} \cos \frac{x}{2} = 2 \sin x \cos \frac{x}{2}$$

$$\text{or } \cos \frac{x}{2} \left[\cos \frac{5x}{2} - \sin x \right] = 0$$

$$\Rightarrow \cos \frac{x}{2} = 0 \text{ or } \cos \frac{5x}{2} = \sin x$$

$$\Rightarrow \frac{x}{2} = 2n\pi \pm \frac{\pi}{2} \text{ or } \frac{5x}{2} = 2n\pi \pm \left(\frac{\pi}{2} - x\right), n \in \mathbb{Z}$$

$$\Rightarrow \frac{x}{2} = (4n \pm 1)\frac{\pi}{2} \text{ or } \frac{5x}{2} = 2n\pi + \frac{\pi}{2} - x, \frac{5x}{2} = 2n\pi - \frac{\pi}{2} + x$$

$$\Rightarrow x = (4n \pm 1)\pi \text{ or } x = \frac{4n\pi}{7} + \frac{\pi}{7}, x = 4n\frac{\pi}{3} - \frac{\pi}{3}$$

Since, $0 < x < 2\pi$,

$$x = \frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$$

21. (1), (4)

$$2(2\sin^2 x - 3\sin x + 1) + 2^3 - (2\sin^2 x - 3\sin x + 1) = 9$$

$$\text{Let } 2(2\sin^2 x - 3\sin x + 1) = t$$

$$\Rightarrow t + \frac{8}{t} = 9 \text{ or } t^2 - 9t + 8 = 0$$

$$\text{or } t = 1, 8$$

$$\Rightarrow 2\sin^2 x - 3\sin x + 1 = 3$$

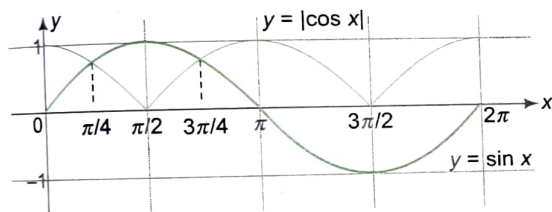
$$\text{or } 2\sin^2 x - 3\sin x + 1 = 0$$

$$\Rightarrow \sin x = -\frac{1}{2}, \sin x = \frac{1}{2}, \sin x = 1$$

22. (1), (3)

It is easier to solve the inequality using graphical method.

The graphs of $y = |\cos x|$ and $y = \sin x$ are as shown in the following figure.



From the figure, $|\cos x| \leq \sin x$ for $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

23. (1), (2)

$$\cos 3\theta + \sin 3\theta + (2\sin 2\theta - 3)(\sin \theta - \cos \theta) > 0$$

$$\text{or } 4(\cos \theta - \sin \theta)(\cos^2 \theta + \sin \theta \cos \theta + \sin^2 \theta - \sin \theta \cos \theta) > 0$$

$$\text{or } -4\sqrt{2} \sin\left(\theta - \frac{\pi}{4}\right) > 0$$

$$\text{or } \sin\left(\theta - \frac{\pi}{4}\right) < 0$$

$$\Rightarrow (2n-1)\pi < \theta - \frac{\pi}{4} < 2n\pi, n \in \mathbb{I}$$

$$\Rightarrow 2n\pi - \frac{3\pi}{4} < \theta < 2n\pi + \frac{\pi}{4}, n \in \mathbb{I}$$

24. (1), (2)

$$1 + (\sin x - \cos x) \sin \frac{\pi}{4} = 2 \cos^2 \frac{5x}{2} \quad \dots(i)$$

$$\Rightarrow 1 + (\sin x - \cos x) \frac{1}{\sqrt{2}} = 1 + \cos 5x$$

$$\Rightarrow \cos 5x + \cos\left(x + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow 2 \cos\left(3x + \frac{\pi}{8}\right) \cos\left(2x - \frac{\pi}{8}\right) = 0$$

$$\Rightarrow \cos\left(3x + \frac{\pi}{8}\right) = 0, \text{ or } \cos\left(2x - \frac{\pi}{8}\right) = 0 \quad \dots(ii)$$

$$3x + \frac{\pi}{8} = n\pi + \frac{\pi}{2} \text{ or } 2x - \frac{\pi}{8} = n\pi + \frac{\pi}{2} \quad \dots(iii)$$

$$\text{or } x = \frac{n\pi}{3} + \frac{\pi}{8} \text{ or } x = \frac{n\pi}{2} + \frac{5\pi}{16}, n \in \mathbb{Z}$$

25. (3), (4)

$(x+y)$ and $(x-y)$ satisfy the equation $\tan^2 \theta - 4 \tan \theta + 1 = 0$.

Thus,

$$\tan(x+y) + \tan(x-y) = 4$$

$$\text{and } \tan(x+y) \tan(x-y) = 1$$

$$\therefore \tan 2x = \tan((x+y) + (x-y))$$

$$\text{or } \tan 2x = \frac{\tan(x+y) + \tan(x-y)}{1 - \tan(x+y) \tan(x-y)}$$

$$\therefore \tan 2x = \infty$$

$$\text{or } 2x = 90^\circ$$

$$\text{or } x = 45^\circ = \frac{\pi}{4}$$

$$\therefore y = \frac{\pi}{6}$$

26. (1), (2), (3), (4)

$$\sin x \sin y = \frac{\sqrt{3}}{4} \text{ and } \cos x \cos y = \frac{\sqrt{3}}{4}$$

$$\text{Then, } \cos x \cos y + \sin x \sin y = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos(x-y) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x-y = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{I} \quad \dots(i)$$

$$\text{and } \cos x \cos y - \sin x \sin y = 0$$

$$\Rightarrow \cos(x+y) = 0$$

$$\Rightarrow x+y = k\pi + \frac{\pi}{2}, k \in \mathbb{I} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$2x = 2n\pi + k\pi \pm \frac{\pi}{6} + \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{2}(2n+k) \pm \frac{\pi}{12} + \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{2}(2n+k) + \frac{\pi}{3} \text{ or } x = \frac{\pi}{2}(2n+k) + \frac{\pi}{6}$$

$$\therefore y = \frac{\pi}{6} + \frac{\pi}{2}(k-2n) \text{ and } y = \frac{\pi}{3} + \frac{\pi}{2}(k-2n)$$

27. (1), (2), (3)

$$f(x) = \left(\cos a_1 + \frac{\cos a_2}{2} + \dots + \frac{\cos a_n}{2^{n-1}} \right) \cos x$$

$$- \left(\frac{\sin a_1}{1} + \frac{\sin a_2}{2} + \dots + \frac{\sin a_n}{2^{n-1}} \right) \sin x$$

$$\Rightarrow f(x) = A \cos x - B \sin x$$

$$\text{Now } f(x_1) = f(x_2) = 0$$

$$\Rightarrow \begin{cases} A \cos x_1 - B \sin x_1 = 0 \\ A \cos x_2 - B \sin x_2 = 0 \end{cases}$$

$$\Rightarrow \tan x_1 = \tan x_2$$

$$\Rightarrow x_1 = n\pi + x_2 \Rightarrow x_1 - x_2 = n\pi$$

28. (1), (3), (4)

The equation becomes

$$(\sin \theta - 2)(\sin \theta + \lambda)(2 \sin \theta + 1) = 0$$

$$\Rightarrow \lambda = \pm 1, 0.$$

29. (2), (3)

If $a = 0$ then $\tan x = 0 \Rightarrow x = n\pi$ and for any value of y such that $\cos y = 0$ the second equation is satisfied.

So, option (1) is not correct.

If $a = 1$ then $\tan x = \cot x \Rightarrow \tan^2 x = 1$ i.e., $\tan 2x$ is not defined.

So, option (2) is correct.

Now, from the first equation, $\tan x = \sqrt{a}$, i.e., a must be positive.

$$\Rightarrow \text{Now, } |\cos y| = \left| \frac{2 \tan x}{b(1 - \tan^2 x)} \right| = \left| \frac{2\sqrt{a}}{b(1 - a)} \right| \leq 1$$

$$\Rightarrow 2\sqrt{a} \leq |b(1 - a)|$$

30. (1), (4)

$$\left(\cos^2 x + \frac{1}{\cos^2 x} \right) (1 + \tan^2 2y) (3 + \sin 3z) = 4$$

$$\text{Since } \cos^2 x + \frac{1}{\cos^2 x} \geq 2, 1 + \tan^2 2y \geq 1$$

and $2 \leq 3 + \sin 3z \leq 4$, the only possibility is

$$\cos^2 x + \frac{1}{\cos^2 x} = 2, 1 + \tan^2 2y = 1, 3 + \sin 3z = 2$$

Thus, $\cos x = \pm 1 \Rightarrow x = n\pi$,

$$\tan 2y = 0 \Rightarrow y = \frac{m\pi}{2} \text{ and}$$

$$\sin 3z = -1$$

$$\Rightarrow z = (4k - 1) \frac{\pi}{6}; m, n, k \in I$$

31. (1), (4)

$$(\tan^2 x + 8 \tan x + 15)(\tan^2 x + 8 \tan x + 7) = 33$$

Let $\tan^2 x + 8 \tan x = p$, where $p \geq -16$

$$(p + 15)(p + 7) = 33$$

$$\Rightarrow p^2 + 22p + 72 = 0$$

$$\Rightarrow p = -18 \text{ (rejected) or } p = -4$$

$$\Rightarrow \tan^2 x + 8 \tan x + 4 = 0$$

$$\text{So, } \tan x + 4 = \pm \sqrt{12}$$

$$\Rightarrow \tan x = -4 + \sqrt{12} \text{ or } -4 - \sqrt{12}$$

Linked Comprehension Type

$$1. (2) \quad x^3 - (1 + \cos \theta + \sin \theta) x^2 + (\cos \theta \sin \theta + \cos \theta + \sin \theta) x - \sin \theta \cos \theta = 0$$

Given cubic function is

$$f(x) = (x - 1)(x - \cos \theta)(x - \sin \theta)$$

Therefore, roots are 1, $\sin \theta$, and $\cos \theta$.

$$\text{Hence, } x_1^2 + x_2^2 + x_3^2 = 1 + \sin^2 \theta + \cos^2 \theta = 2$$

2. (3) Now if $1 = \sin \theta$, we get $\theta = \pi/2$

If $1 = \cos \theta$, then $\theta = 0, 2\pi$ and if $\sin \theta = \cos \theta$, we get $\tan \theta = 1$. Therefore,

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

Therefore, the number of values of θ in $[0, 2\pi]$ is 5.

3. (1) Again the maximum possible difference between the two roots is 2.

$$1 - \sin \theta = 2, \text{ when } \theta = 3\pi/2$$

$$\text{or } 1 - \cos \theta = 2, \text{ when } \theta = \pi$$

4. (1), 5. (3), 6. (4)

We have equation $\sec x + \operatorname{cosec} x = a$

To analyze the roots of the equation, we draw the graph of function $y = \sec x + \operatorname{cosec} x$ and check how many times line $y = a$ intersects this graph.

Period of $y = \sec x + \operatorname{cosec} x$ is 2π .

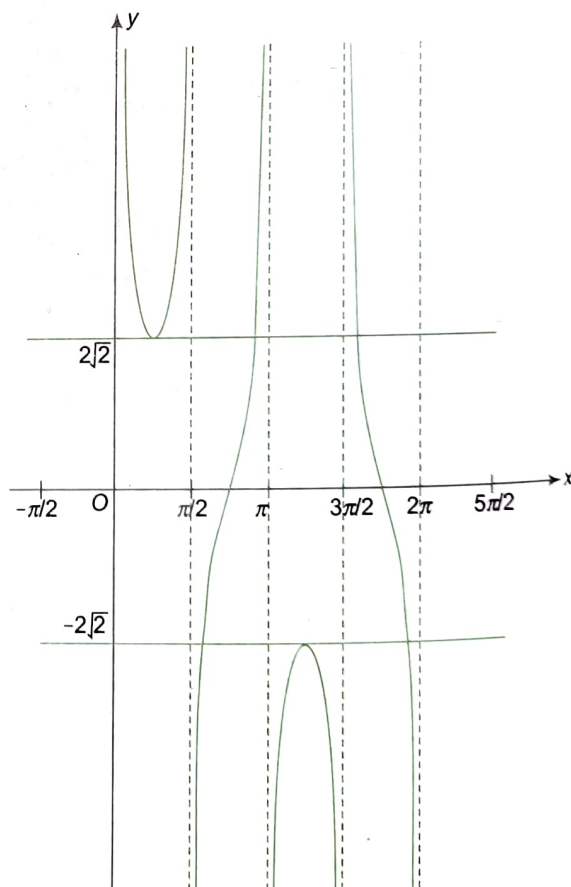
So, we draw the graph of the function for $x \in [0, 2\pi]$.

The graph of function can be easily drawn by drawing the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ and then adding the values of $\sec x$ and $\operatorname{cosec} x$ by inspection.

For example, in first quadrant, $\sec x, \operatorname{cosec} x > 0$.

Also, when x approaches to zero, $\operatorname{cosec} x$ approaches to infinity.

So, $f(x)$ approaches to infinity.



Similarly, when x approaches to $\pi/2$, $\sec x$ approaches to infinity.

So, $f(x)$ approaches to infinity.

At $x = \pi/4$, $f(x)$ attains its least value which is $2\sqrt{2}$.

With similar arguments, we can draw the graph of $y = f(x)$ in intervals $(\pi/2, \pi)$, $(\pi, 3\pi/2)$ and $(3\pi/2, 2\pi)$.

We have following graph of $y = f(x)$.

From the figure, we can say that $f(x) = a$ has two distinct solutions if line $y = a$ cuts the graph $y = f(x)$ between $y = 2\sqrt{2}$ and $y = -2\sqrt{2}$, i.e., $|a| < 2\sqrt{2}$.

If line $y = a$, cuts the graph of $y = f(x)$ above $y = 2\sqrt{2}$ and below $y = -2\sqrt{2}$, then $f(x) = a$ has four distinct solutions. So, $|a| > 2\sqrt{2}$.

8. (2), 9. (4)

The given system is

$$\sin x \cos 2y = (a^2 - 1)^2 + 1,$$

$$\text{and } \cos x \sin 2y = a + 1$$

...(i)

Since the L.H.S. of both the equations does not exceed 1, the given system may have solutions only for a 's such that

$$(a^2 - 1)^2 + 1 \leq 1 \text{ and } -1 \leq a + 1 \leq 1$$

...(ii)

$$(a^2 - 1)^2 + 1 \leq 1$$

$$\text{or } (a^2 - 1)^2 \leq 0$$

$$\text{or } (a^2 - 1)^2 = 0$$

$$\text{or } a = 1$$

For $a = 1$, equation $\cos x \sin 2y = a + 1$ does not hold.

Thus, $a = -1$ only and we get

$$\sin x \cos 2y = 1$$

$$\cos x \sin 2y = 0$$

$$\sin x \cos 2y = 1$$

$$\Rightarrow \sin x = 1, \cos 2y = 1$$

$$\text{or } \sin x = -1, \cos 2y = -1$$

for which $\cos x \sin 2y = 0$.

10. (1), 11. (4), 12. (2)

$$\text{Given that } \int_0^x (t^2 - 8t + 13) dt = x \sin\left(\frac{a}{x}\right)$$

...(i)

For R.H.S., $x \neq 0$

Integrating L.H.S., we get

$$\left[\frac{t^3}{3} - 4t^2 + 13t \right]_0^x = x \sin\left(\frac{a}{x}\right)$$

$$\text{or } \left(\frac{1}{3} \right) [x^3 - 12x^2 + 39x] = x \sin\left(\frac{a}{x}\right)$$

$$\text{or } \sin\left(\frac{a}{x}\right) = \left(\frac{1}{3} \right) [x^2 - 12x + 39] \quad \{ \because x \neq 0 \}$$

$$= \left(\frac{1}{3} \right) \{ (x-6)^2 + 3 \}$$

$$= \left(\frac{1}{3} \right) (x-6)^2 + 1$$

But $\sin(a/x) \leq 1$, so $\sin(a/x) = 1$, which is possible only for $x = 6$

Then we have $\sin(a/6) = 1$

$$\text{or } a/6 = 2n\pi + \pi/2 \text{ or } a = 12n\pi + 3\pi, n \in \mathbb{Z}$$

Hence, $x = 6$, $a = 12n\pi + 3\pi$, $n \in \mathbb{Z}$.

For $a \in [0, 100]$, there are exactly three values of $a = 3\pi, 15\pi$, and 27π , i.e.,

$$|y - \cos a| < x$$

$$\Rightarrow |y + 1| < 6$$

$$\Rightarrow y \in [-7, 5]$$

13. (1), 14. (4), 15. (2)

The given equations are

$$x \cos^3 y + 3x \cos y \sin^2 y = 14 \quad \dots(i)$$

$$\text{and } x \sin^3 y + 3x \cos^2 y \sin y = 13 \quad \dots(ii)$$

Adding Eqs. (i) and (ii), we have

$$x (\cos^3 y + 3 \cos y \sin^2 y + 3 \cos^2 y \sin y + \sin^3 y) = 27$$

$$\text{or } x (\cos y + \sin y)^3 = 27$$

$$\text{or } x^{1/3} (\cos y + \sin y) = 3 \quad \dots(iii)$$

Subtracting Eq. (ii) from Eq. (i), we have

$$x (\cos^3 y + 3 \cos y \sin^2 y - 3 \cos^2 y \sin y - \sin^3 y) = 1$$

$$\text{or } x (\cos y - \sin y)^3 = 1$$

$$\text{or } x^{1/3} (\cos y - \sin y) = 1 \quad \dots(iv)$$

Dividing Eq. (iii) by (iv), we get

$$\cos y + \sin y = 3 \cos y - 3 \sin y$$

$$\text{or } \tan y = 1/2$$

Case I:

$$\sin y = 1/\sqrt{5} \text{ and } \cos y = 2/\sqrt{5}$$

$$y = 2n\pi + \alpha, \text{ where } 0 < \alpha < \pi/2 \text{ and } \sin \alpha = 1/\sqrt{5}$$

i.e., y lies in the first quadrant

$$\text{From Eqs. (iii), } x^{1/3} (3/\sqrt{5}) = 3 \text{ or } x = 5\sqrt{5}$$

Case II:

$$\sin y = -1/\sqrt{5} \text{ and } \cos y = -2/\sqrt{5}$$

$$y = 2n\pi + (\pi + \alpha), \text{ where } 0 < \alpha < \pi/2$$

$$\text{and } \sin \alpha = 1/\sqrt{5}$$

i.e., y lies in the third quadrant.

$$\text{Therefore, from Eq. (iii), } x^{1/3} (-3/\sqrt{5}) = 3 \text{ or } x = -5\sqrt{5}.$$

$$\text{Thus, } \sin^2 y + 2 \cos^2 y = 1/5 + 8/5 = 9/5.$$

Also there are exactly six values of $y \in [0, 6\pi]$, three in 1st quadrant and three in 3rd quadrant.

$$16. (1) (1 + a) \cos \theta \cos (2\theta - b) = (1 + a \cos 2\theta) \cos (\theta - b)$$

$$\Rightarrow \cos \theta \cos (2\theta - b) + a \cos \theta \cos (2\theta - b)$$

$$= \cos (\theta - b) + a \cos 2\theta \cos (\theta - b).$$

$$\Rightarrow \cos (3\theta - b) + \cos (\theta - b) + a \{ \cos (3\theta - b) + \cos (\theta - b) \}$$

$$= 2 \cos (\theta - b) + a \{ \cos (3\theta - b) + \cos (\theta + b) \}$$

$$\Rightarrow \cos (3\theta - b) - \cos (\theta - b) = a \cos (\theta + b) - a \cos (\theta - b)$$

$$\Rightarrow 2 \sin (2\theta - b) \sin \theta = 2a \sin \theta \sin b$$

$$\Rightarrow \sin \theta = 0 \text{ or } \sin (2\theta - b) = a \sin b$$

$$\text{If } \sin \theta = 0 \text{ then } \theta = n\pi, n \in \mathbb{Z} \Rightarrow S_1 = \{ n\pi | n \in \mathbb{Z} \}$$

$$\text{Also, } 2\theta - b = n\pi + (-1)^n \sin^{-1}(a \sin b), n \in \mathbb{Z}$$

$$\Rightarrow S_2 = \frac{1}{2} \{ n\pi + (-1)^n \sin^{-1}(a \sin b) + b, n \in \mathbb{Z} \}$$

$$17. (3) \sin (2\theta - b) = a \sin b \Rightarrow |a \sin b| \leq 1$$

$$18. (2) a = 0 \Rightarrow \sin(2\theta - b) = 0.$$

$$\Rightarrow 2\theta - b = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow \theta = \frac{n\pi + b}{2}, n \in \mathbb{Z}; S_2 = \frac{n\pi + b}{2}, n \in \mathbb{Z}$$

S_2 is subset of $(0, \pi)$

$$\therefore 0 < \frac{n\pi + b}{2} < \pi, n \in \mathbb{Z}$$

$$\therefore 0 < n\pi + b < 2\pi \Rightarrow -n\pi < b < -n\pi + 2\pi$$

$$\Rightarrow b \in (-n\pi, 2\pi - n\pi), n \in \mathbb{Z}$$

19. (1), 20. (3)

$$\frac{b \cos x}{2 \cos 2x - 1} = \frac{b + \sin x}{(\cos^2 x - 3 \sin^2 x) \tan x}$$

$$(1) \quad 2 \cos 2x - 1 \neq 0 \Rightarrow x \neq n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

$$(2) \quad \tan x \neq 0 \Rightarrow x \neq n\pi, n \in \mathbb{Z}$$

$$(3) \quad \cos^2 x - 3 \sin^2 x \neq 0 \Rightarrow x \neq n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

$$\text{Also } 2 \cos 2x - 1 = 2(\cos^2 x - \sin^2 x) - (\cos^2 x + \sin^2 x) \\ = \cos^2 x - 3 \sin^2 x$$

Now, the given equation reduces to

$$b \sin x = b + \sin x \\ \Rightarrow \sin x = \frac{b}{b-1}$$

$$\text{Now, } -1 \leq \sin x \leq 1$$

$$\Rightarrow -1 \leq \frac{b}{b-1} \leq 1$$

$$\Rightarrow \frac{b}{b-1} + 1 \geq 0 \text{ and } \frac{b}{b-1} - 1 \leq 0$$

$$\Rightarrow \frac{2b-1}{b-1} \geq 0 \text{ and } \frac{1}{b-1} \leq 0$$

$$\Rightarrow b \in \left(-\infty, \frac{1}{2}\right] \cup (1, \infty) \text{ and } b \in (-\infty, 1)$$

$$\sin x \neq 0, \pm \frac{1}{2} \text{ or } b \neq 0, -1, \frac{1}{3}$$

When $b = \frac{1}{2}$ then $\sin x = -1$, which is not possible, as $\tan x$ is not defined for this value of $\sin x$.

$$\text{So, } b \in \left(-\infty, \frac{1}{2}\right) - \left\{-1, 0, \frac{1}{3}\right\}$$

For any other value of b , $\sin x$ takes two values for $x \in (0, \pi)$.

Matrix Match Type

1. $a \rightarrow r, b \rightarrow s, c \rightarrow p, d \rightarrow q.$

$$a. \cos^2 2x - \sin^2 x = 0$$

$$\Rightarrow \cos 3x \cos x = 0$$

$$\Rightarrow \cos 3x = 0 \text{ or } \cos x = 0$$

$$\Rightarrow 3x = (2n-1) \frac{\pi}{2} \text{ or } x = (2n-1) \frac{\pi}{2}, n \in \mathbb{Z}$$

$$\text{or } x = (2n-1) \frac{\pi}{6} \text{ or } x = (2n-1) \frac{\pi}{2}$$

Hence, the general solution is $(2n-1) \frac{\pi}{6}$ as $(2n-1) \frac{\pi}{2}$ is contained in $(2n-1) \frac{\pi}{6}$.

$$b. \cos x + \sqrt{3} \sin x = \sqrt{3}$$

$$\text{or } \frac{\cos x}{2} + \frac{\sqrt{3}}{2} \sin x = \frac{\sqrt{3}}{2}$$

$$\text{or } \cos\left(x - \frac{\pi}{3}\right) = \cos \frac{\pi}{6}$$

$$\Rightarrow x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2} \text{ or } x = 2n\pi + \frac{\pi}{6}$$

$$c. \sqrt{3} \tan^2 x - (\sqrt{3} + 1) \tan x + 1 = 0$$

$$\text{or } \sqrt{3} \tan x (\tan x - 1) - (\tan x - 1) = 0$$

$$\text{or } (\tan x - 1)(\sqrt{3} \tan x - 1) = 0$$

$$\therefore \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\text{or } \tan x = \frac{1}{\sqrt{3}} \Rightarrow x = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$$

$$d. \tan 3x - \tan 2x - \tan x = 0$$

$$\text{or } \tan x \tan 2x \tan 3x = 0$$

$$x = n\pi \text{ or } n\pi/2 \text{ (rejected) or } n\pi/3$$

Therefore, the general solution is $n\pi/3$ as $n\pi$ is contained in $n\pi/3$.

2. $a \rightarrow q; b \rightarrow s; c \rightarrow s; d \rightarrow p.$

$$a. \text{ Here, } x^3 + (x+2)^2 + 2 \sin x = 4.$$

Clearly, $x = 0$ satisfies the equation

$$\text{If } 0 < x \leq \pi, x^3 + (x+2)^2 + 2 \sin x > 4$$

$$\text{If } \pi < x \leq 2\pi, x^3 + (x+2)^2 + 2 \sin x > 27 + 25 - 2$$

So, $x = 0$ is the only solution.

$$b. \text{ Here, } \frac{1}{2} \sin(2e^x) = \frac{1}{4} (2^x + 2^{-x}) \geq \frac{1}{4} \cdot 2 = \frac{1}{2}$$

($\because$ A.M. $\geq$ G.M.)

$$\text{or } \sin(2e^x) \geq 1 \text{ or } \sin(2e^x) = 1$$

But equality can hold when

$$2^x = 2^{-x} = 1, \text{ i.e., } x = 0.$$

Then $\sin(2 \times e^0) = 1$, which is not true.

Hence, there is no solution.

$$c. \sin 2x + \cos 4x = 2$$

$$\Rightarrow \sin 2x = 1, \cos 4x = 1$$

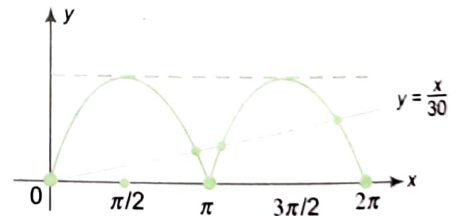
$$\therefore 1 - 2 \sin^2 2x = 1 \text{ or } 1 - 2 = 1 \text{ (absurd)}$$

$$d. \text{ The given solution is } |\sin x| = x/30.$$

Therefore, the solution of this equation is the point of intersections of the curves.

$$y = |\sin x| \text{ and } y = x/30$$

Since there are four points of intersection in $0 \leq x \leq 2\pi$ the equation has four solutions.



3. $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r.$

$$a. 5 \sin \theta + 3 (\sin \theta \cos \alpha - \cos \theta \sin \alpha)$$

$$= (5 + 3 \cos \alpha) \sin \theta - 3 \sin \alpha \cos \theta$$

$$\Rightarrow \max_{\theta \in \mathbb{R}} \{5 \sin \theta + 3 \sin(\theta - \alpha)\}$$

$$= \sqrt{(5 + 3 \cos \alpha)^2 + 9 \sin^2 \alpha}$$

$$= \sqrt{34 + 30 \cos \alpha}$$

Therefore, the given equation is

$$34 + 30 \cos \alpha = 49$$

$$\text{or } \cos \alpha = 1/2$$

$$\Rightarrow \alpha = 2n\pi \pm \pi/3, n \in \mathbb{Z}$$

$$h. (\cos x)^{\sin^2 x - 3 \sin x + 2} = 1$$

$$\Rightarrow \sin^2 x - 3 \sin x + 2 = 0$$

$$\Rightarrow (\sin x - 2)(\sin x - 1) = 0$$

$$\Rightarrow \sin x = 1$$

But this does not satisfy the equation because $0^0 = 1$ is absurd.

$$c. \sqrt{(\sin x)} + 2^{1/4} \cos x = 0 \quad \dots(i)$$

$$\therefore \sqrt{(\sin x)} > 0 \text{ and so } \cos x < 0,$$

Also $\sin x > 0 \Rightarrow x$ lies in the second quadrant

Equation (i) can be rewritten as

$$2^{1/4} \cos x = -\sqrt{(\sin x)}$$

$$\text{On squaring, we get } \sqrt{2} \cos^2 x = \sin x$$

$$\text{or } \sqrt{2} \sin^2 x + \sin x - \sqrt{2} = 0$$

$$\text{or } (\sqrt{2} \sin x - 1)(\sin x + \sqrt{2}) = 0$$

$$\sin x \neq -\sqrt{2}, \text{ therefore } \sin x = 1/\sqrt{2} \quad \dots(ii)$$

$$\Rightarrow x = 3\pi/4 \text{ and so the general value of } x \text{ is given by}$$

$$x = 2n\pi + 3\pi/4, n \in \mathbb{Z}$$

$$d. \log_5 \tan x = (\log_5 4)(\log_4 (3 \sin x))$$

$$\Rightarrow \log_5 \tan x = \log_5 (3 \sin x) \quad \dots(i)$$

Since $\log x$ is real when $x > 0$, we have

$$\tan x > 0, \sin x > 0$$

Therefore, x lies in the first quadrant. Now Eq. (i) gives

$$\tan x = 3 \sin x \text{ or } \cos x = 1/3$$

$$\therefore x = 2n\pi + \cos^{-1}(1/3), n \in \mathbb{Z}$$

$$4.(1) a. \cos ax + \cos bx = 2, (a \neq 0, b \neq 0)$$

$$\Rightarrow ax = 2n\pi \text{ and } bx = 2m\pi, m, n \in \mathbb{Z}$$

$$\Rightarrow \frac{a}{b} = \frac{n}{m} \quad (\text{here } \frac{a}{b} \text{ must be rational})$$

or LCM of a and b must exist. Also, $x = 0$ is one solution
So, equation has at least one solution.

$$b. \cos x + \cos(\sqrt{2})x = 2$$

$$\Rightarrow x = 2n\pi \text{ and } \sqrt{2}x = 2m\pi, m, n \in \mathbb{Z}$$

$$\Rightarrow \sqrt{2}n = m, \text{ which is satisfied by } m = n = 0 \text{ only.}$$

So, equation has only one solution $x = 0$.

LCM of $(1, \sqrt{2})$ does not exist or $\frac{1}{\sqrt{2}}$ is irrational number.

$$c. \cos ax + \cos bx = 1, (a \neq 0, b \neq 0)$$

Clearly, equation has infinite solutions.

$$d. \cos x + \cos \pi x = -2$$

$$\Rightarrow \cos x = -1 \text{ and } \cos \pi x = -1$$

$$\Rightarrow x = (2k+1)\pi \text{ and } \pi x = (2m+1)\pi, m, n \in \mathbb{Z}$$

$$\Rightarrow (2k+1)\pi = (2m+1)\pi; \text{ which does not hold.}$$

So, equation has no solution.

$$5.(1) \sin^2 x + (2a-3)\sin x + (a^2-3a+2) = 0$$

$$\Rightarrow \sin x = 1-a, 2-a$$

Equation has four solutions if

$$-1 < 1-a < 1 \text{ and } -1 < 2-a < 1$$

$$\Rightarrow 0 < a < 2 \text{ and } 1 < a < 3$$

$$\Rightarrow 1 < a < 2$$

Equation has three solutions if

$$-a+1 = -1 \text{ and } -a+2 < 1$$

$$\text{or } -a+2 = 1 \text{ and } -a+1 > -1$$

$$\Rightarrow a = 1, 2$$

Equation has two solutions if

$$-1 < 1-a < 1 \text{ and } -a+2 > 1$$

$$\text{or } 1-a < -1 \text{ and } -1 < 2-a < 1$$

$$\Rightarrow a \in (0, 1) \cup (2, 3)$$

Equation has one solution if

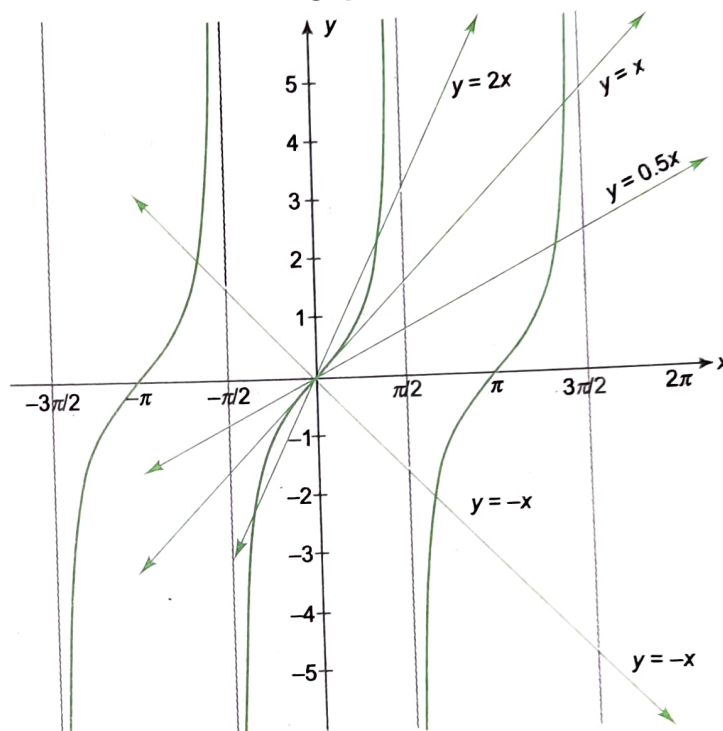
$$1-a = \pm 1 \text{ and } 2-a > 1$$

$$\Rightarrow a = 0$$

$$\text{or } 2-a = \pm 1 \text{ and } 1-a < -1$$

$$\Rightarrow a = 3$$

6. (4) Graphs of $y = \tan x$, $y = -x$, $y = 0.5x$, $y = x$ and $y = 2x$ are as shown in the following figure.



From the figure,

$\tan x = -x$ has least positive root in $(\pi/2, \pi)$;

$\tan x = 0.5x$ and $\tan x = x$ have least positive roots in $(\pi, 3\pi/2)$;

and $\tan x = 2x$ has least positive root in $(0, \pi/2)$.

Numerical Value Type

$$1. (1) \sin^3 x + p^3 + 1 = 3p \sin x$$

$$\Rightarrow (\sin x + p + 1)(\sin^2 x + 1 + p^2 - \sin x - p - p \sin x) = 0$$

$$\text{Therefore, either } \sin x + p + 1 = 0 \Rightarrow p = -(1 + \sin x)$$

$$\text{or } \sin x = 1 = p$$

Hence, only one value of $p(p > 0)$ is possible which is given by $p = 1$.

$$2. (2) \log_{0.5} \sin x = 1 - \log_{0.5} \cos x, x \in [-2\pi, 2\pi]$$

$$\sin x > 0 \text{ and } \cos x > 0$$

$$\sin x \cos x = 1/2$$

$$\therefore \sin 2x = 1$$

Therefore, there are 2 values of x .

$$3. (3) 4 \leq \text{L.H.S.} \leq 16$$

$$2 \leq \text{R.H.S.} \leq 4$$

Hence, equality can occur only when both sides are 4, which is possible if $x = \pi, 3\pi, 5\pi$.

That is, there are three solutions.

$$4. (6) \quad \frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = 0$$

$$\text{or} \quad \frac{2\sin x \cos x}{2\cos 3x \cos x} + \frac{2\sin 3x \cos 3x}{2\cos 9x \cos 3x} + \frac{2\sin 9x \cos 9x}{2\cos 27x \cos 9x} = 0$$

$$\text{or} \quad \frac{\sin(3x-x)}{2\cos 3x \cos x} + \frac{\sin(9x-3x)}{2\cos 9x \cos 3x} + \frac{\sin(27x-9x)}{2\cos 27x \cos 9x} = 0$$

$$\text{or} \quad (\tan 3x - \tan x) + (\tan 9x - \tan 3x) + (\tan 27x - \tan 9x) = 0$$

$$\text{or} \quad \tan 27x - \tan x = 0$$

$$\text{or} \quad \tan x = \tan 27x$$

$$\Rightarrow 27x = n\pi + x, n \in I$$

$$\text{or} \quad x = \frac{n\pi}{26}, n \in I.$$

$$\text{or} \quad x = \frac{\pi}{26}, \frac{2\pi}{26}, \frac{3\pi}{26}, \frac{4\pi}{26}, \frac{5\pi}{26}, \frac{6\pi}{26}$$

Hence, there are six solutions.

$$5. (1) \quad (\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = 2^{3x} = (2\sqrt{2})^{2x}$$

$$\Rightarrow \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2x} + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^{2x} = 1$$

$$\Rightarrow (\sin 75^\circ)^{2x} + (\cos 75^\circ)^{2x} = 1$$

$$\Rightarrow x = 1$$

$$6. (4) \quad \text{Since } -2 \leq \sin x - \sqrt{3} \cos x \leq 2, \text{ we have}$$

$$-2 \leq \frac{4m-6}{4-m} \leq 2$$

$$\text{or} \quad -1 \leq \frac{2m-3}{4-m} \leq 1$$

$$\text{If } \frac{2m-3}{4-m} \leq 1, \text{ we have}$$

$$\frac{(2m-3)-(4-m)}{4-m} \leq 0$$

$$\text{or} \quad \frac{3m-7}{m-4} \geq 0 \quad \dots(i)$$

$$\text{Also, } -1 \leq \frac{2m-3}{4-m}$$

$$\Rightarrow \frac{m+1}{m-4} \leq 0 \quad \dots(ii)$$

$$\text{From Eqs. (i) and (ii), we get } m \in \left[-1, \frac{7}{3}\right]$$

Therefore, the possible integers are $-1, 0, 1, 2$.

$$7. (2) \quad \text{We have, } \left(\cos\left(x + \frac{\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right)\right)^2 = \sin^2 x$$

$$\Rightarrow 4 \sin^2\left(\frac{x}{2}\right) \cdot \sin^2\left(\frac{x}{2} + \frac{\pi}{6}\right) = 4 \sin^2\left(\frac{x}{2}\right) \cdot \cos^2\left(\frac{x}{2}\right)$$

$$\therefore \sin \frac{x}{2} = 0 \text{ or } \sin^2\left(\frac{x}{2} + \frac{\pi}{6}\right) = \cos^2 \frac{x}{2}$$

$$\Rightarrow x = 2m\pi, m \in Z$$

$$\text{or} \quad \frac{\pi}{2} - \left(\frac{x}{2} + \frac{\pi}{6}\right) = n\pi \pm \frac{x}{2}, n \in I$$

$$\therefore x = 0 \text{ or } x = \frac{\pi}{3}$$

$$\begin{aligned} 8. (5) \quad \cos 4x &= 2 \cos^2 2x - 1 \\ &= 2(2 \cos^2 x - 1)^2 - 1 \\ &= 2(4 \cos^4 x + 1 - 4 \cos^2 x) - 1 \\ &= 8 \cos^4 x - 8 \cos^2 x + 1 \end{aligned}$$

$$\therefore a_0 = 1, a_1 = -8, a_2 = 8$$

$$\therefore 5a_0 + a_1 + a_2 = 5$$

$$9. (1) \quad 1 - \sin^2 x - \sin x + a = 0$$

$$\Rightarrow \sin^2 x + \sin x = (a+1)$$

For $x \in (0, \pi/2)$, the range of $\sin^2 x + \sin x$ is $(0, 2)$ (i)

$$\Rightarrow 0 < (a+1) < 2$$

$$\Rightarrow a \in (-1, 1)$$

$$\begin{aligned} 10. (0) \quad \frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x} &= \frac{\frac{1}{2}(\sin x - \cos x)^2}{\cos^2 x - \sin^2 x} \\ &= \frac{-\frac{1}{2}(\sin x - \cos x)}{\cos x + \sin x} = -\frac{1}{2} \tan\left(x - \frac{\pi}{4}\right) \end{aligned}$$

Given equation reduces to

$$2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{-\frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)} + 1 = 0$$

$$\Rightarrow 2^{\tan\left(x - \frac{\pi}{4}\right)} = 1$$

$\Rightarrow x = \pi/4$, which is not possible as $\cos 2x = 0$ for this value of x , which is not defining the original equation.

$$11. (4) \quad \sin^4 x - \cos^2 x \sin x + 2 \sin^2 x + \sin x = 0$$

$$\text{or} \quad \sin x [\sin^3 x - \cos^2 x + 2 \sin x + 1] = 0$$

$$\text{or} \quad \sin x [\sin^3 x + \sin^2 x + 2 \sin x] = 0$$

$$\text{or} \quad \sin^2 x [\sin^2 x + \sin x + 2] = 0$$

$$\text{or} \quad \sin x = 0, \text{ where } x = 0, \pi, 2\pi, 3\pi$$

Hence, there are four solutions.

$$12. (5) \quad 3 \cot^2 x + 8 \cot x + 3 = 0$$

$$\therefore \cot x = \frac{-4 \pm \sqrt{7}}{3}$$

Both roots are negative.

$$\therefore \frac{\pi}{2} < x_1, x_2 < \pi$$

$$\therefore \pi < x_1 + x_2 < 2\pi$$

Product of roots $= \cot x_1 \cot x_2 = 1$

$$\Rightarrow \cot x_1 \cot\left(\frac{3\pi}{2} - x_1\right) = 1$$

$$\Rightarrow x_1 + x_2 = \frac{3\pi}{2}$$

$$\text{Similarly, } x_1 + x_2 = \frac{7\pi}{2}$$

$$13. (6) \quad 2 \sin \theta = r^4 - 2r^2 + 3$$

$$\Rightarrow 2 \sin \theta = (r^2 - 1)^2 + 2$$

This is possible only when $\sin \theta = 1$ and $r^2 = 1$ or $r = \pm 1$.

So, $\theta = \pi/2, 5\pi/2, 9\pi/2$

$\therefore$ Number of values of the pair $(r, \theta) = 6$

$$14. (7) \quad \text{Here } (2 \sec x + 1)(\sec x - 3) = 0 \text{ which gives two values of } \theta \text{ in each of } [0, 2\pi], (2\pi, 4\pi], (4\pi, 6\pi] \text{ and one value in } \left(6\pi, 6\pi + \frac{3\pi}{2}\right).$$

$$\cos \alpha \cos x - \sin x \leq \sin y \quad \forall x \in R$$

$$\therefore \sin y \geq \text{greatest value of } (\cos \alpha \cos x - \sin x)$$

$$\therefore \sin y \geq \sqrt{\cos^2 \alpha + 1}$$

$$\therefore \sin y = 1 \text{ and } \cos \alpha = 0$$

$$\therefore \sin(\sin x + \cos x) = \cos(\cos x - \sin x)$$

$$\text{or } \cos(\cos x - \sin x) = \cos\left(\frac{\pi}{2} - (\sin x + \cos x)\right)$$

$$\text{or } \cos x - \sin x = 2n\pi \pm \left(\frac{\pi}{2} - \sin x - \cos x\right)$$

Taking +ve sign, we get

$$\cos x - \sin x = 2n\pi + \frac{\pi}{2} - \sin x - \cos x$$

$$\Rightarrow \cos x = n\pi + \frac{\pi}{4}$$

For $n = 0$, $\cos x = \frac{\pi}{4}$, which is the only possible value

$$\therefore \sin x = \frac{(\sqrt{16 - \pi^2})}{4} \quad \dots(i)$$

Taking -ve sign, we get

$$\sin x = \frac{\pi}{4} \quad \dots(ii)$$

From (i) and (ii), we get $\frac{\pi}{4}$ as the largest value.

Hence, $k = 4$.

$$17. (2) \text{ Given } \frac{\pi}{2} < \left| 3x - \frac{\pi}{2} \right| \leq \pi$$

$$\Rightarrow \frac{\pi}{2} < \left(3x - \frac{\pi}{2} \right) \leq \pi$$

$$\text{or } -\pi \leq \left(3x - \frac{\pi}{2} \right) < \frac{-\pi}{2}$$

$$\Rightarrow x \in \left[\frac{-\pi}{6}, 0 \right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2} \right]$$

$$\text{Now, } 1 + \cos x + \cos 2x + \sin x + \sin 2x + \sin 3x = 0$$

$$\Rightarrow 2 \cos^2 x + \cos x + \sin 2x + 2 \sin 2x \cos x = 0$$

$$\Rightarrow \cos x(2 \cos x + 1) + \sin 2x(2 \cos x + 1) = 0$$

$$\Rightarrow (\cos x + \sin 2x)(2 \cos x + 1) = 0$$

$$\Rightarrow \cos x(1 + 2 \sin x)(2 \cos x + 1) = 0$$

$$\Rightarrow \cos x = 0 \text{ or } \sin x = \frac{-1}{2} \text{ (as for given interval, } \cos x > 0)$$

$$\Rightarrow x = \frac{\pi}{2} \text{ or } x = \frac{-\pi}{6}$$

Hence, there are 2 solutions.

$$18. (14) \text{ We have } \sqrt{a-3} \sin 2x - \sqrt{a} (1 - 2 \sin^2 x) = 5$$

$$\therefore \sqrt{a-3} \sin 2x - \sqrt{a} \cos 2x = 5$$

This equation has at least one solution if

$$5 \leq \sqrt{(\sqrt{a-3})^2 + (\sqrt{a})^2}$$

$$\text{or } 2a - 3 \geq 25$$

$$\therefore a \geq 14$$

So, least value of a is 14.

$$19. (12) \text{ We have } \sin^2(x+y) = \sin^2(x-y)$$

$$\therefore x+y = n\pi \pm (x-y)$$

$$\Rightarrow x = \frac{n\pi}{2}, y \in R$$

$$\text{or } y = \frac{n\pi}{2}, x \in R$$

Thus, for each of $x = 0, \pm \frac{\pi}{2}$, there are two values of y .

For each of $y = 0, \pm \frac{\pi}{2}$, there are two values of x .

So, total number of ordered pairs is 12.

$$20. (8) \quad x \left(\sin^2 x + \frac{1}{x^2} \right) = 2 \sin x \sin^2 y$$

$$\Rightarrow \sin^2 x + \frac{1}{x^2} = 2 \left(\frac{\sin x}{x} \right) \sin^2 y$$

$$\Rightarrow \sin^2 x + \frac{1}{x^2} \leq 2 \left(\frac{\sin x}{x} \right)$$

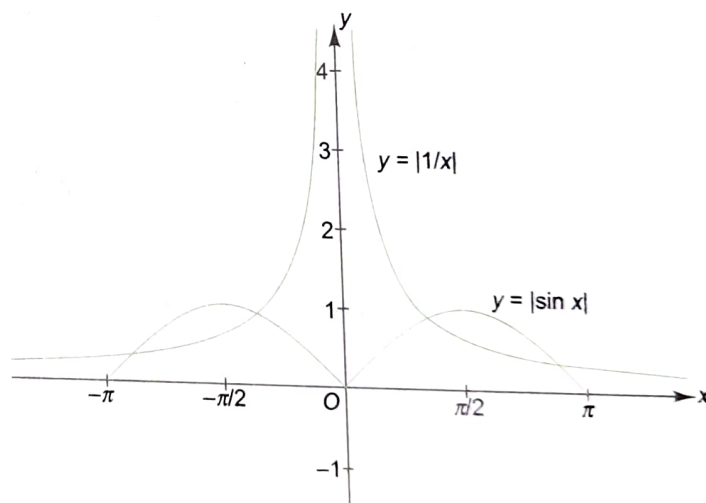
But using A.M. $\geq$ G.M., we have

$$\sin^2 x + \frac{1}{x^2} \geq 2 \frac{\sin x}{x}$$

$$\Rightarrow \sin^2 x = \frac{1}{x^2}$$

$$\text{or } |\sin x| = \left| \frac{1}{x} \right|$$

Draw the graph of $y = |\sin x|$ and $y = \left| \frac{1}{x} \right|$ as shown in the following figure.



From the figure, we get the four values of x .

$$\sin^2 y = 1 \text{ gives } y = \frac{\pi}{2}, \frac{3\pi}{2}$$

So, there will be $4 \times 2 = 8$ ordered pairs.

$$21. (13) \text{ We have } \cos 5x \times \tan 6|x| + \sin 5x = 0$$

Case I:

$$x \geq 0$$

$$\therefore \cos 5x \tan 6x + \sin 5x = 0$$

$$\Rightarrow \cos 5x \sin 6x + \sin 5x \cos 6x = 0$$

$$\Rightarrow \sin 11x = 0$$

$$\Rightarrow 11x = n\pi$$

$$\Rightarrow x = \frac{n\pi}{11}, n \in \mathbb{Z}^+ \cup \{0\}$$

So, there will be eleven solutions for $n = 0, 1, 2, \dots, 10$.

Case II:

$$x < 0$$

$$\therefore -\cos 5x \tan 6x + \sin 5x = 0$$

$$\Rightarrow -\cos 5x \sin 6x + \cos 6x \sin 5x = 0$$

$$\Rightarrow \sin(5x - 6x) = 0$$

$$\Rightarrow \sin x = 0$$

$$\Rightarrow x = n\pi, n \in \mathbb{Z}$$

$$\therefore x = -\pi, -2\pi$$

So, total number of solutions is 13.

Archives

JEE MAIN

Single Correct Answer Type

1. (2) $\cos x + \cos 2x + \cos 3x + \cos 4x = 0, 0 \leq x < 2\pi$

$$\Rightarrow (\cos x + \cos 4x) + (\cos 2x + \cos 3x) = 0$$

$$\Rightarrow 2 \cos \frac{5x}{2} \cos \frac{3x}{2} + 2 \cos \frac{5x}{2} \cos \frac{x}{2} = 0$$

$$\Rightarrow 2 \cos \frac{5x}{2} \left[2 \cos x \cos \frac{x}{2} \right] = 0$$

$$\Rightarrow \cos \frac{5x}{2} = 0$$

$$\text{or } \cos x = 0 \quad \text{or } \cos \frac{x}{2} = 0$$

$$\Rightarrow x = \frac{(2n+1)\pi}{5} \quad \text{or } x = (2n+1) \frac{\pi}{2}$$

$$\text{or } x = (2n+1)\pi, n \in \mathbb{Z}$$

$$\Rightarrow x = \left\{ \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{2} \right\}$$

$\therefore$ Number of solution is 7

2. (3) $8 \cos x \left(\cos \left(\frac{\pi}{6} + x \right) \cos \left(\frac{\pi}{6} - x \right) - \frac{1}{2} \right) = 1$

$$\Rightarrow 8 \cos x \left[\cos^2 \frac{\pi}{6} - \sin^2 x - \frac{1}{2} \right] = 1$$

$$\Rightarrow 8 \cos x \left[\frac{1}{4} - (1 - \cos^2 x) \right] = 1$$

$$\Rightarrow 8 \cos^3 x - 6 \cos x = 1$$

$$\Rightarrow 4 \cos^3 x - 3 \cos x = \frac{1}{2}$$

$$\Rightarrow \cos 3x = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$3x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

Since $x \in [0, \pi]$, possible values are

$$3x = \frac{\pi}{3}, 2\pi \pm \frac{\pi}{3}$$

$$\therefore x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

$$\text{Sum of solutions} = \frac{13\pi}{9}$$

$$\therefore k = \frac{13}{9}$$

JEE ADVANCED

Single Correct Answer Type

1. (1) $(f \circ g \circ f)(x) = \sin^2(\sin x^2)$

$$(g \circ f)(x) = \sin(\sin x^2)$$

$$\therefore \sin^2(\sin x^2) = \sin(\sin x^2)$$

$$\Rightarrow \sin(\sin x^2)[\sin(\sin x^2) - 1] = 0$$

$$\Rightarrow \sin(\sin x^2) = 0 \text{ or } 1$$

$$\Rightarrow \sin x^2 = n\pi \text{ or } 2m\pi + \pi/2, \text{ where } m, n \in \mathbb{Z}$$

$$\Rightarrow \sin x^2 = 0$$

$$\Rightarrow x^2 = n\pi \Rightarrow x = \pm \sqrt{n\pi}, n \in \{0, 1, 2, \dots\}$$

2. (4) $\sin x + 2 \sin 2x - \sin 3x = 3$

$$\Rightarrow \sin x + 4 \sin x \cos x - 3 \sin x + 4 \sin^3 x = 3$$

$$\Rightarrow \sin x [-2 + 4 \cos x + 4(1 - \cos^2 x)] = 3$$

$$\Rightarrow \sin x [2 - (4 \cos^2 x - 4 \cos x + 1) + 1] = 3$$

$$\Rightarrow 3 - (2 \cos x - 1)^2 = 3 \operatorname{cosec} x$$

Now R.H.S. ≥ 3

But L.H.S. < 3

Hence, no solution.

3. (3) $\sqrt{3} \sec x + \operatorname{cosec} x = 2(\cot x - \tan x)$

$$\Rightarrow \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x} = 2 \left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} \right)$$

$$\Rightarrow \sqrt{3} \sin x + \cos x = 2(\cos^2 x - \sin^2 x)$$

$$\Rightarrow \cos \left(x - \frac{\pi}{3} \right) = \cos 2x$$

$$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3} \right), n \in \mathbb{Z}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{3}, \frac{2n\pi}{3} + \frac{\pi}{9}$$

$$\text{In } (-\pi, \pi), x = \frac{-\pi}{3}, \frac{\pi}{9}, \frac{7\pi}{9}, \frac{-5\pi}{9}$$

$$\therefore \sum x_i = 0$$

Multiple Correct Answers Type

1. (3), (4)

$$\sum_{m=1}^6 \operatorname{cosec} \left(\theta + \frac{(m-1)\pi}{4} \right) \operatorname{cosec} \left(\theta + \frac{m\pi}{4} \right) = 4\sqrt{2}$$

$$\Rightarrow \frac{1}{\sin(\pi/4)} \left[\frac{\sin(\theta + \pi/4 - \theta)}{\sin \theta \cdot \sin(\theta + \pi/4)} + \frac{\sin((\theta + \pi/2) - (\theta + \pi/4))}{\sin(\theta + \pi/4) \cdot \sin(\theta + \pi/2)} + \dots + \frac{\sin((\theta + 3\pi/2) - (\theta + 5\pi/4))}{\sin(\theta + 3\pi/2) \cdot \sin(\theta + 5\pi/4)} \right] = 4\sqrt{2}$$

$$\Rightarrow \frac{1}{\sin(\pi/4)} \left[\frac{\sin(\theta + \pi/4) \cos \theta - \cos(\theta + \pi/4) \sin \theta}{\sin \theta \cdot \sin(\theta + \pi/4)} + \frac{\sin(\theta + \pi/2) \cos(\theta + \pi/4)}{\sin(\theta + \pi/4) \cdot \sin(\theta + \pi/2)} - \frac{\cos(\theta + \pi/2) \sin(\theta + \pi/4)}{\sin(\theta + \pi/4) \cdot \sin(\theta + \pi/2)} + \dots + \frac{\sin(\theta + 3\pi/2) \cos(\theta + 5\pi/4)}{\sin(\theta + 3\pi/2) \cdot \sin(\theta + 5\pi/4)} - \frac{\cos(\theta + 3\pi/2) \sin(\theta + 5\pi/4)}{\sin(\theta + 3\pi/2) \cdot \sin(\theta + 5\pi/4)} \right] = 4\sqrt{2}$$

$$\Rightarrow \sqrt{2} [\cot \theta - \cot(\theta + \pi/4) + \cot(\theta + \pi/4) - \cot(\theta + \pi/2) + \dots + \cot(\theta + 5\pi/4) - \cot(\theta + 3\pi/2)] = 4\sqrt{2}$$

$$\Rightarrow \tan \theta + \cot \theta = 4$$

$$\Rightarrow \tan \theta = 2 \pm \sqrt{3}$$

$$\Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

∴ (1), (3), (4)

$$2 \cos \theta (1 - \sin \phi) = \frac{2 \sin^2 \theta}{\sin \theta} \cos \phi - 1$$

$$= 2 \sin \theta \cos \phi - 1$$

$$\therefore 2 \cos \theta - 2 \cos \theta \sin \phi = 2 \sin \theta \cos \phi - 1$$

$$\therefore 2 \cos \theta + 1 = 2 \sin(\theta + \phi)$$

$$\therefore \tan(2\pi - \theta) > 0$$

$$\Rightarrow \tan \theta < 0$$

$$\text{and } -1 < \sin \theta < -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$$

$$\Rightarrow 0 < \cos \theta < \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} < \sin(\theta + \phi) < 1$$

$$\Rightarrow \frac{\pi}{6} + 2\pi < \theta + \phi < \frac{5\pi}{6} + 2\pi$$

$$2\pi + \frac{\pi}{6} - \theta_{\max} < \phi < 2\pi + \frac{5\pi}{6} - \theta_{\min}$$

$$\Rightarrow \frac{\pi}{2} < \phi < \frac{4\pi}{3}$$

... (i)

(from (i))

Matrix Match Type

1. (a) → (q), (s)

$$\text{We have } 2\sin^2 \theta + \sin^2 2\theta = 2$$

$$\Rightarrow 2\sin^2 \theta + 4\sin^2 \theta \cos^2 \theta = 2$$

$$\Rightarrow \sin^2 \theta + 2\sin^2 \theta (1 - \sin^2 \theta) = 1$$

$$\Rightarrow 3\sin^2 \theta - 2\sin^4 \theta - 1 = 0$$

$$\Rightarrow \sin \theta = \pm \frac{1}{\sqrt{2}}, \pm 1$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{2}$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (3) We have

$$(y+z) \cos 3\theta = (xyz) \sin 3\theta \quad \dots (i)$$

$$xyz \sin 3\theta = (2 \cos 3\theta)z + (2 \sin 3\theta)y \quad \dots (ii)$$

$$(xyz) \sin 3\theta = (y+2z) \cos 3\theta + y \sin 3\theta \quad \dots (iii)$$

$$\therefore (y+z) \cos 3\theta = (2 \cos 3\theta)z + (2 \sin 3\theta)y$$

$$= (y+2z) \cos 3\theta + y \sin 3\theta$$

$$\Rightarrow y(\cos 3\theta - 2 \sin 3\theta) = z \cos 3\theta$$

$$\text{and } y(\sin 3\theta - \cos 3\theta) = 0$$

$$\text{Since } y, z \neq 0, \sin 3\theta - \cos 3\theta = 0 \text{ or } \tan 3\theta = 0$$

$$\Rightarrow 3\theta = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\Rightarrow \theta = \frac{(4n+1)\pi}{12}, n \in \mathbb{Z}$$

$$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$$

2. (3) $\tan \theta = \cot 5\theta$

$$\Rightarrow \cos 6\theta = 0$$

$$\Rightarrow 4 \cos^3 2\theta - 3 \cos 2\theta = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or } \pm \frac{\sqrt{3}}{2} \quad (i)$$

$$\sin 2\theta = \cos 4\theta$$

$$\Rightarrow 2 \sin^2 2\theta + \sin 2\theta - 1 = 0$$

$$\Rightarrow (2 \sin 2\theta - 1)(\sin 2\theta + 1) = 0$$

$$\Rightarrow \sin 2\theta = -1 \text{ or } \sin 2\theta = \frac{1}{2} \quad \dots (ii)$$

From (i) and (ii)

$$\cos 2\theta = 0 \text{ and } \sin 2\theta = -1$$

$$\Rightarrow 2\theta = -\frac{\pi}{2}$$

$$\Rightarrow \theta = -\frac{\pi}{4}$$

$$\text{or } \cos 2\theta = \pm \frac{\sqrt{3}}{2}, \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\therefore \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$$

Linked Comprehension Type

1. (3) $f(x) = (1-x)^2 \sin^2 x + x^2 \quad \forall x \in \mathbb{R}$

For statement P:

$$f(x) + 2x = 2(1+x^2)$$

$$\Rightarrow (1-x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$$

$$\Rightarrow (1-x)^2 \sin^2 x = x^2 - 2x + 2 = (x-1)^2 + 1$$

$$\Rightarrow (1-x)^2 (\sin^2 x - 1) = 1$$

$$\Rightarrow -(1-x)^2 \cos^2 x = 1$$

$$\Rightarrow (1-x)^2 \cos^2 x = -1$$

So equation (i) will not have real solution.

So, P is wrong.

For statement Q:

$$2(1-x)^2 \sin^2 x + 2x^2 + 1 = 2x + 2x^2$$

$$2(1-x)^2 \sin^2 x = 2x - 1$$

$$2 \sin^2 x = \frac{2x-1}{(1-x)^2}$$

$$\text{Let } h(x) = \frac{2x-1}{(1-x)^2} - 2 \sin^2 x$$

$$\text{Clearly, } h(0) = -1, \lim_{x \rightarrow 1^-} h(x) = +\infty$$

So by IVT, equation (ii) will have solution.

So, Q is correct.

$$\begin{aligned}
 3. (8) \quad & \frac{5}{4} \cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2 \\
 \Rightarrow & \frac{5}{4} \cos^2 2x + (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x + \\
 & (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x) = 2 \\
 \Rightarrow & \frac{5}{4} \cos^2 2x + 1 - \frac{1}{2} \sin^2 2x + 1 - \frac{3}{4} \sin^2 2x = 2 \\
 \Rightarrow & \cos^2 2x = \sin^2 2x \\
 \Rightarrow & \tan^2 2x = 1 \Rightarrow \tan 2x = \pm 1 \\
 \Rightarrow & 2x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z} \\
 \Rightarrow & x = (4n \pm 1) \frac{\pi}{8}, n \in \mathbb{Z} \\
 \Rightarrow & x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}
 \end{aligned}$$

So, number of solutions = 8.

$$4. (0.5) \text{ We have } \sqrt{3}a \cos x + 2b \sin x = c, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \sqrt{3}a \left(\frac{1-t^2}{1+t^2} \right) + 2b \left(\frac{2t}{1+t^2} \right) = c, \text{ where } t = \tan \frac{x}{2}$$

$$\begin{aligned}
 \Rightarrow & \sqrt{3}a(1-t^2) + 4bt = c(1+t^2) \\
 \Rightarrow & t^2(c + \sqrt{3}a) - 4bt + c - \sqrt{3}a = 0
 \end{aligned}$$

Equation has roots $\tan \frac{\alpha}{2}, \tan \frac{\beta}{2}$.

$$\text{So, } \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = \frac{4b}{c + \sqrt{3}a}$$

$$\text{and } \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{c - \sqrt{3}a}{c + \sqrt{3}a}$$

$$\text{Now, } \frac{\alpha + \beta}{2} = \frac{\pi}{6}$$

$$\Rightarrow \tan \left(\frac{\alpha + \beta}{2} \right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{4b}{c + \sqrt{3}a - c + \sqrt{3}a} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{b}{a} = \frac{1}{2}$$

Chapter 5

Concept Application Exercises

Exercise 5.1

1. Let $\angle A = \frac{\pi}{2} \Rightarrow a^2 = b^2 + c^2$ and $2R = a$

$$\Rightarrow \frac{a^2 + b^2 + c^2}{R^2} = \frac{2a^2}{R^2} = \frac{2a^2 \times 4}{a^2} = 8$$

2. Angles are in A.P.; so, $B = \frac{\pi}{3}$

Also, $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$ or $\frac{b}{\sqrt{3}} = \frac{c}{\sqrt{2}}$ or $\frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1}{\sqrt{2}}}$

But we know that $\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A}$

$$\therefore B = \frac{\pi}{3}, C = \frac{\pi}{4} \Rightarrow A = \frac{5\pi}{12}$$

3. We have $(\sqrt{3} - 1)a = 2b$

$$\Rightarrow \frac{a}{2} = \frac{b}{\sqrt{3} - 1}$$

$$\Rightarrow \frac{\sin A}{2} = \frac{\sin B}{\sqrt{3} - 1}$$

$$\Rightarrow \frac{\sin 3B}{2} = \frac{\sin B}{\sqrt{3} - 1}$$

$$\Rightarrow 3 - 4 \sin^2 B = \frac{2}{\sqrt{3} - 1}$$

$$\Rightarrow 3 - 4 \sin^2 B = \sqrt{3} + 1$$

$$\Rightarrow \sin^2 B = \frac{2 - \sqrt{3}}{4}$$

$$\Rightarrow \sin^2 B = \frac{(\sqrt{3} - 1)^2}{8}$$

$$\Rightarrow \sin B = \pm \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\Rightarrow \sin B = \frac{\sqrt{3} - 1}{2\sqrt{2}} \quad (\text{as } \sin B \text{ is positive})$$

Thus, $B = 15^\circ, A = 45^\circ$ and $C = 120^\circ$.

4. $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$

or $\frac{\cos A}{2R \sin A} = \frac{\cos B}{2R \sin B} = \frac{\cos C}{2R \sin C}$

or $\tan A = \tan B = \tan C$

Hence, triangle is equilateral.

Therefore, Area of triangle = $\frac{\sqrt{3}}{4} a^2 = \sqrt{3}$ (as $a = 2$)

5. $\cos^2 A + \cos^2 B - \cos^2 C = 1$

or $1 - \sin^2 A + 1 - \sin^2 B - 1 + \sin^2 C = 1$

or $\sin^2 A + \sin^2 B = \sin^2 C \Rightarrow a^2 + b^2 = c^2$

Thus, the triangle is right angled at C.

6. $b^2 \cos 2A - a^2 \cos 2B$

$$= b^2(1 - 2 \sin^2 A) - a^2(1 - 2 \sin^2 B)$$

$$= b^2 - a^2 - 2(b^2 \sin^2 A - a^2 \sin^2 B) = b^2 - a^2$$

$$(\because a \sin B = b \sin A)$$

7. $\frac{c}{a+b} = \frac{2R \sin C}{2R(\sin A + \sin B)} = \frac{\sin(A+B)}{\sin A + \sin B}$

$$= \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$= \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}$$

$$= \frac{\cos \frac{1}{2} A \cos \frac{1}{2} B - \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B}$$

$$= \frac{1 - \tan \frac{1}{2} A \tan \frac{1}{2} B}{1 + \tan \frac{1}{2} A \tan \frac{1}{2} B}$$

[dividing numerator and

denominator by $\cos \frac{1}{2} A \cos \frac{1}{2} B]$

8. Since $a = 2R \sin A, b = 2R \sin B$, and $c = 2R \sin C$, we have

$$(b^2 - c^2) \cot A = 4R^2 (\sin^2 B - \sin^2 C) \cot A$$

$$= 4R^2 \sin(B+C) \sin(B-C) \cot A$$

$$= 4R^2 \sin A \sin(B-C) \frac{\cos A}{\sin A}$$

$$= -4R^2 \sin(B-C) \cos(B+C)$$

$$(\because \cos A = -\cos(B+C))$$

$$= -2R^2 [2 \sin(B-C) \cos(B+C)]$$

$$= -2R^2 [\sin 2B - \sin 2C] \quad (i)$$

Similarly, $(c^2 - a^2) \cot B = -2R^2 [\sin 2C - \sin 2A]$ (ii)

and $(a^2 - b^2) \cot C = -2R^2 [\sin 2A - \sin 2B]$ (iii)

Adding Eqs. (i), (ii), and (iii), we get

$$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$$

9. In the triangle ABC,

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A}$$

$$= \frac{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}}$$

$$= \frac{\cos \frac{B-C}{2}}{\sin \frac{A}{2}}$$

$$\leq \frac{1}{\sin \frac{A}{2}}$$

$$\Rightarrow \frac{b+c}{a} \leq \operatorname{cosec} \frac{A}{2}$$

$$\begin{aligned}
 10. \quad \frac{1 + \cos(A-B)\cos C}{1 + \cos(A-C)\cos B} &= \frac{1 - \cos(A-B)\cos(A+B)}{1 - \cos(A-C)\cos(A+C)} \\
 &= \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)} \\
 &= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} \\
 &= \frac{a^2 + b^2}{a^2 + c^2} \quad [\text{using } a = 2R \sin A, \text{ etc.}]
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \frac{b}{c} \sin 2C + \frac{c}{b} \sin 2B + \frac{b}{a} \sin 2A + \frac{a}{b} \sin 2B &= 2 \\
 \Rightarrow 2 \sin B \cos C + 2 \sin C \cos B + 2 \sin B \cos A &+ 2 \sin B \cos B = 2 \\
 \Rightarrow \sin(B+C) + \sin(A+B) &= 1 \\
 \Rightarrow \sin A + \sin C &= 1 \\
 \Rightarrow \sin B &= \frac{1}{2} \quad (\text{As } 2 \sin B = \sin A + \sin C)
 \end{aligned}$$

$$\begin{aligned}
 12. \quad a \cos A + b \cos B + c \cos C &= R \sin 2A \\
 \Rightarrow a \cos A + b \cos B + c \cos C &= R(\sin 2A + \sin 2B + \sin 2C) \\
 &= 4R \sin A \sin B \sin C
 \end{aligned}$$

Exercise 5.2

1. Sides are $a, b, \sqrt{a^2 + ab + b^2}$

Then the greatest side $c = \sqrt{a^2 + ab + b^2}$

Let the angle opposite to the greatest side be C .

$$\Rightarrow \cos C = \frac{a^2 + b^2 - (a^2 + ab + b^2)}{2ab} = \frac{-1}{2}$$

$$\Rightarrow C = 120^\circ$$

2. Here $(AB)^2 = (a-c)^2 + (b-d)^2$
 $(OA)^2 = (a-0)^2 + (b-0)^2 = a^2 + b^2$
 and $(OB)^2 = c^2 + d^2$
 B(c, d)

O $\searrow^\theta$ A(a, b)

Now in $\triangle AOB$,

$$\begin{aligned}
 \cos \theta &= \frac{(OA)^2 + (OB)^2 - (AB)^2}{2OA \times OB} \\
 &= \frac{a^2 + b^2 + c^2 + d^2 - [(a-c)^2 + (b-d)^2]}{2\sqrt{a^2 + b^2} \sqrt{c^2 + d^2}} \\
 &= \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}
 \end{aligned}$$

3. Let $a = 3x + 4y$, $b = 4x + 3y$, and $c = 5x + 5y$

Obviously, c is the greatest side; hence, $\angle C$ is the greatest angle.

Thus,

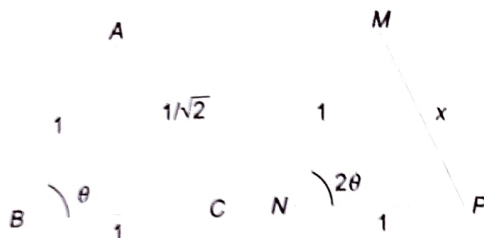
$$\begin{aligned}
 \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\
 &= \frac{-2xy}{2(12x^2 + 25xy + 12y^2)} < 0
 \end{aligned}$$

Since $\cos C < 0$, $\angle C > \pi/2$. So, triangle is obtuse angled.

4. Given $a + b = 20$ and $c + a = 21$
 Now, $a^2 = b^2 + c^2 - 2bc \cos(120^\circ)$

$$\begin{aligned}
 \Rightarrow a^2 &= (20-a)^2 + (21-a)^2 + \frac{2(20-a)(21-a)}{2} \\
 \text{or } 2a^2 - 123a + 1261 &= 0 \\
 \text{or } 2a^2 - 26a - 97a + 1261 &= 0 \\
 \text{or } 2a(a-13) - 97(a-13) &= 0 \\
 \text{or } a &= 13, 97/2 \\
 \therefore a &= 13 \quad [\text{as } a = 97/2 \text{ is not possible}]
 \end{aligned}$$

5.



$$\text{In } \triangle ABC, \cos \theta = \frac{1+1-\frac{1}{2}}{2} = \frac{3}{4}$$

$$\Rightarrow \cos 2\theta = 2 \cos^2 \theta - 1 = 2 \times \frac{9}{16} - 1 = \frac{1}{8}$$

$$\begin{aligned} \text{In } \triangle MNP, x^2 &= 1+1-2 \cos 2\theta \\ &= 2(1 - \cos 2\theta) = 2 \left(1 - \frac{1}{8}\right) = \frac{7}{4} \end{aligned}$$

$$\text{or } x = \frac{\sqrt{7}}{2}$$

6. We have $\frac{bc}{2 \cos A} = b^2 + c^2 - 2bc \cos A = a^2$

$$\Rightarrow \cos A = \frac{bc}{2a^2}$$

$$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{bc}{2a^2}$$

$$\Rightarrow b^2 c^2 = a^2 (b^2 + c^2 - a^2)$$

$$\Rightarrow (a^2 - b^2)(a^2 - c^2) = 0$$

$$\Rightarrow a = b \text{ or } a = c$$

Hence, triangle is isosceles.

7. Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = k$

$$\therefore b + c = 11k$$

$$c + a = 12k$$

$$a + b = 13k$$

Adding the above three equations, we get

$$2(a + b + c) = 36k$$

$$\text{or } a + b + c = 18k$$

From Eqs. (i) and (iv), $a = 7k$

From Eqs. (ii) and (iv), $b = 6k$

From Eqs. (iii) and (iv), $c = 5k$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36k^2 + 25k^2 - 49k^2}{2 \times 6k \times 5k}$$

$$= \frac{12k^2}{60k^2} = \frac{1}{5}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{25k^2 + 49k^2 - 36k^2}{2 \times 5k \times 7k}$$

$$= \frac{38k^2}{70k^2} = \frac{19}{35}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49k^2 + 36k^2 - 25k^2}{2 \times 7k \times 6k}$$

$$= \frac{60k^2}{84k^2} = \frac{5}{7}$$

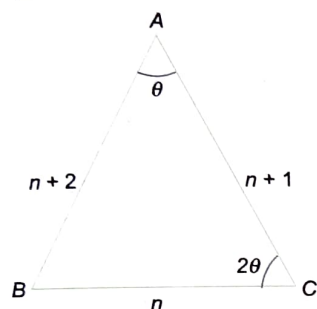
$$\therefore \frac{\cos A}{(1/5)} = \frac{\cos B}{(19/35)} = \frac{\cos C}{(5/7)}$$

$$\text{or } \frac{\cos A}{(7/35)} = \frac{\cos B}{(19/35)} = \frac{\cos C}{(25/35)}$$

$$\text{or } \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

Let $a = n$, $b = n + 1$, $c = n + 2$, $n \in \mathbb{N}$

Let the smallest angle $\angle A = \theta$, then the greatest angle $\angle C = 2\theta$.



In $\triangle ABC$, by applying the sine law, we get

$$\frac{\sin \theta}{n} = \frac{\sin 2\theta}{n+2}$$

$$\text{or } \frac{\sin \theta}{n} = \frac{2 \sin \theta \cos \theta}{n+2}$$

$$\text{or } \frac{1}{n} = \frac{2 \cos \theta}{n+2} \quad [\text{as } \sin \theta \neq 0]$$

$$\Rightarrow \cos \theta = \frac{n+2}{2n} \quad (i)$$

In $\triangle ABC$, by the cosine law, we get

$$\cos \theta = \frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)} \quad (ii)$$

Comparing the value of $\cos \theta$ from Eqs. (i) and (ii), we get

$$\frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)} = \frac{n+2}{2n}$$

$$\text{or } (n+2)^2(n+1) = n(n+2)^2 + n(n+1)^2 - n^3$$

$$\text{or } n(n+2)^2 + (n+2)^2 = n(n+2)^2 + n(n+1)^2 - n^3$$

$$\text{or } n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$$

$$\text{or } n^2 - 3n - 4 = 0$$

$$\text{or } (n+1)(n-4) = 0$$

$$\text{or } n = 4 \quad [\text{as } n \neq -1]$$

Therefore, the sides of the triangle are 4, 4 + 1, 4 + 2, i.e., 4, 5, 6.

Exercise 5.3

$$1. c \cos(A - \alpha) + a \cos(C + \alpha)$$

$$= c(\cos A \cos \alpha + \sin A \sin \alpha) + a(\cos C \cos \alpha - \sin C \sin \alpha)$$

$$= \cos \alpha (c \cos A + a \cos C) + c \sin A \sin \alpha - a \sin C \sin \alpha$$

$$= b \cos \alpha + \sin \alpha (c \sin A - a \sin C) = b \cos \alpha$$

$$2. \frac{\cos C + \cos A}{c + a} + \frac{\cos B}{b}$$

$$= \frac{(b \cos C + c \cos B) + (b \cos A + a \cos B)}{b(c + a)}$$

$$= \frac{a + c}{b(c + a)} = \frac{1}{b}$$

$$3. ab^2 \cos A + ba^2 \cos B + ac^2 \cos A + ca^2 \cos C$$

$$+ bc^2 \cos B + b^2 c \cos C$$

$$= ab(b \cos A + a \cos B) + ac(c \cos A + a \cos C)$$

$$+ bc(c \cos B + b \cos C)$$

$$= abc + abc + abc = 3abc$$

Exercise 5.4

$$1. \text{ Given } b + c = 3a \quad \dots(i)$$

$$\cot \frac{B}{2} \cot \frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{s}{s-a} = \frac{2s}{2s-2a}$$

$$= \frac{a+b+c}{b+c-a} = \frac{4a}{2a} = 2 \quad [\text{by Eq. (i)}]$$

$$2. \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\text{or } bc \cos^2 \frac{A}{2} = s(s-a)$$

$$\text{or } bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2}$$

$$= s(3s - (a + b + c)) = s(3s - 2s) = s^2$$

$$3. \tan \frac{A}{2} = \frac{5}{6}, \tan \frac{C}{2} = \frac{2}{5}$$

$$\Rightarrow \tan \frac{A}{2} \tan \frac{C}{2} = \frac{s-b}{s} = \frac{5}{6} \times \frac{2}{5} = \frac{1}{3}$$

$$\text{or } 3s - 3b = s$$

$$\text{or } 2s = 3b$$

$$\text{or } a + b + c = 3b$$

$$\text{or } a + c = 2b$$

Hence, a, b, c are in A.P.

$$4. (b+c-a) \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right)$$

$$= (2s-2a) \left(\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \right)$$

$$= 2(s-a) \sqrt{s} \left(\frac{s-b+s-c}{\sqrt{(s-a)(s-b)(s-c)}} \right)$$

$$= 2(s-a) \sqrt{s} \left(\frac{a}{\sqrt{(s-a)(s-b)(s-c)}} \right)$$

$$= 2a \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= 2a \cot \frac{A}{2}$$

$$5. \sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}, \sin^2 \frac{C}{2} \text{ are in H.P.}$$

$$\Rightarrow \frac{(s-b)(s-c)}{bc}, \frac{(s-a)(s-c)}{ac}, \frac{(s-a)(s-b)}{ab}$$

are in H.P.

$$\Rightarrow \frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c} \text{ are in H.P.}$$

$$\Rightarrow \frac{s-a}{a}, \frac{s-b}{b}, \frac{s-c}{c} \text{ are in A.P.}$$

$$\Rightarrow \frac{s}{a} - 1, \frac{s}{b} - 1, \frac{s}{c} - 1 \text{ are in A.P.}$$

So, a, b and c must be in H.P.

Exercise 5.5

1. $c^2 = a^2 + b^2$

Triangle is right angled at vertex C. Hence,

$$\text{Area of } \Delta = \frac{1}{2}ab$$

$$\text{Also, } \Delta^2 = s(s-a)(s-b)(s-c) = \frac{1}{4}a^2b^2$$

$$\text{or } 4s(s-a)(s-b)(s-c) = a^2b^2$$

2. Let $a = 3k, b = 7k, c = 8k$. Then,

$$s = \frac{1}{2}(a+b+c) = 9k$$

$$\Rightarrow \frac{R}{r} = \frac{abc}{4\Delta} \cdot \frac{s}{\Delta} = \frac{abcs}{4s(s-a)(s-b)(s-c)}$$

$$= \frac{3k \times 7k \times 8k}{4 \times 6k \times 2k \times k} = \frac{7}{2}$$

3. $\Delta = \frac{1}{2}bc \sin A = \frac{9}{2} \sin A = \frac{9}{2} \frac{a}{2R} = \frac{9}{2R}$

$$\Rightarrow R = \frac{9}{2\Delta}$$

4. $BE = 12$ and $CF = 9$

$$\therefore BG = 8 \text{ and } CG = 6$$

$$\text{Area of } \Delta BGC = \frac{1}{2} \times 8 \times 6 \times \sin \theta$$

$$= 24 \sin \theta$$

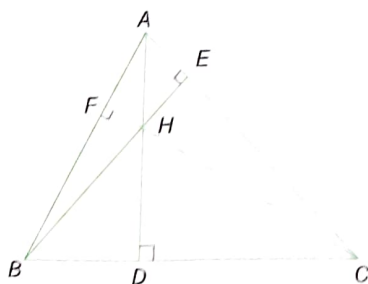
$$Ar(\Delta ABC) = 3 Ar(\Delta BGC)$$

$$= 72 \sin \theta$$

$$\therefore Ar(\Delta ABC)_{\max} = 72 \text{ sq. units}$$

5. Given that $a = \frac{2\Delta}{h_A}; b = \frac{2\Delta}{h_B}; c = \frac{2\Delta}{h_C}$,

$$\text{i.e., } a = \Delta; b = \Delta; c = \frac{2\Delta}{3}$$



$$\Rightarrow a + b + c = 2\Delta \left(\frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} \right)$$

$$\text{or } 2s = 2\Delta \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3} \right) = 2\Delta \left(\frac{8}{6} \right)$$

$$\text{or } s = \frac{4}{3}\Delta$$

...(i)

$$\text{Now } \Delta^2 = s(s-a)(s-b)(s-c)$$

$$\text{or } \Delta^2 = \frac{4}{3} \times \Delta \left(\frac{\Delta}{3} \right) \left(\frac{\Delta}{3} \right) \left(\frac{2\Delta}{3} \right)$$

$$\text{or } 1 = \frac{8}{81} \times \Delta^2$$

$$\text{or } \Delta = \frac{9}{\sqrt{8}}$$

6. The only possible set of sides can be 3, 3, 2 (as sum of any two sides must be greater than the third one)

$$\text{So, } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{4 \times 1 \times 1 \times 2} = 2\sqrt{2} \text{ cm}^2$$

7. Let the sides be $x-d, x, x+d$. Then

$$s = \frac{3x}{2}, (s-a) = \frac{x}{2} + d,$$

$$(s-b) = \frac{x}{2}, (s-c) = \frac{x}{2} - d.$$

$$\text{Area of triangle} = \sqrt{\frac{3x}{2} \cdot \left(\frac{x}{2} + d \right) \cdot \frac{x}{2} \cdot \left(\frac{x}{2} - d \right)}$$

$$= \frac{x}{2} \sqrt{3 \left(\frac{x^2}{4} - d^2 \right)} = \frac{x}{4} \sqrt{3(x^2 - 4d^2)}$$

The area of equilateral triangle whose perimeter is $3x$ is $\frac{\sqrt{3}x^2}{4}$

$$\text{Given, } \frac{3}{5} \times \frac{\sqrt{3}x^2}{4} = \frac{x}{4} \sqrt{3(x^2 - 4d^2)}$$

$$\text{or } \frac{9}{25} \times \frac{3x^4}{16} = \frac{x^2}{16} \times 3(x^2 - 4d^2)$$

$$\text{or } x^2 - \frac{9x^2}{25} = 4d^2$$

$$\text{or } 16x^2 = 100d^2$$

$$\text{or } x = \frac{5}{2}d$$

$$\text{Therefore, the sides of triangle measure } \left(\frac{5d}{2} - d \right), \frac{5d}{2}, \left(\frac{5d}{2} + d \right)$$

$$\text{or } \frac{3d}{2}, \frac{5d}{2}, \frac{7d}{2}$$

Hence, the ratio of sides is 3 : 5 : 7.

For the greatest angle,

$$\cos \theta = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = \frac{9 + 25 - 49}{30} = \frac{-1}{2}$$

$$\therefore \theta = 120^\circ$$

Exercise 5.6

1. We have $\frac{\sin A}{a} = \frac{\sin B}{b}$

$$\Rightarrow a \sin B = b \sin A$$

(a) $b \sin A = a \Rightarrow a \sin B = a$

Since $A < \pi/2$, the ΔABC is possible.

(b) $b \sin A > a \Rightarrow a \sin B > a \Rightarrow \sin B > 1$

Which is impossible. Hence, the possibility (ii) is ruled out.

(c) $b \sin A > a, A < \pi/2$

$$\Rightarrow a \sin B > a$$

$$\Rightarrow \sin B > 1$$

(which is not possible)

(d) $b \sin A < a \Rightarrow a \sin B < a \Rightarrow \sin B < 1$

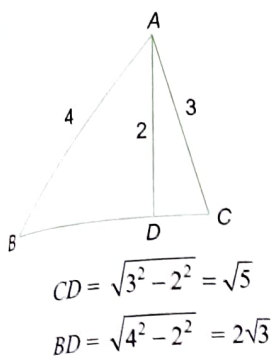
So, value of $\angle B$ exists.

Now, $b > a \Rightarrow B > A$. Since $A < \pi/2$

The $\triangle ABC$ is possible when $B > \pi/2$.

(e) Since $b = a$, we have $B = A$. But $A > \pi/2$

Therefore, $B > \pi/2$. But this is not possible for any triangle.



$$\text{Now, area of } \triangle ABC = \frac{1}{2} \cdot 2 \cdot (\sqrt{5} + 2\sqrt{3}) = \frac{1}{2} \cdot 4 \cdot 3 \cdot \sin A$$

$$\Rightarrow \sin A = \frac{\sqrt{5} + 2\sqrt{3}}{6} < 1$$

Also $a > b, a > c$

Hence, A can have two values.

$$3. \text{ We have } \frac{a}{2} = \frac{b}{3} = k \text{ and}$$

$$\sec^2 A = \frac{8}{5}$$

$$\Rightarrow \cos^2 A = \frac{5}{8}$$

$$\Rightarrow \frac{5}{8} = \left(\frac{9k^2 + c^2 - 4k^2}{6kc} \right)^2 = \left(\frac{5k^2 + c^2}{6kc} \right)^2$$

$$\Rightarrow 45k^2 c^2 = 50k^4 + 20k^2 c^2 + 2c^4$$

$$\Rightarrow 2c^4 - 25k^2 c^2 + 50k^4 = 0$$

$$\Rightarrow c^2 = \frac{25k^2 \pm \sqrt{625k^4 - 400k^4}}{4}$$

$$= \frac{25k^2 \pm 15k^2}{4} = 10k^2, \frac{5}{2} k^2$$

There are two possible valid values of c^2 . Hence there exist two triangle satisfying the given conditions.

4. Given that the angles of triangle are in A.P.

$$\text{Let } \angle A = x - d, \angle B = x, \angle C = x + d$$

$$\text{Now, } \angle A + \angle B + \angle C = 180^\circ$$

$$\Rightarrow x - d + x + x + d = 180^\circ$$

$$\Rightarrow 3x = 180^\circ$$

$$\Rightarrow x = 60^\circ$$

$$\therefore \angle B = 60^\circ$$

$$\text{Using cosine formula } \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ we get}$$

$$\cos 60^\circ = \frac{100 + c^2 - 81}{2 \times 10 \times c}$$

$$\Rightarrow \frac{1}{2} = \frac{19 + c^2}{2 \times 10c}$$

$$\Rightarrow c^2 - 10c + 19 = 0$$

$$\Rightarrow c = \frac{10 \pm \sqrt{100 - 76}}{2}$$

$$\Rightarrow c = 5 \pm \sqrt{6}$$

Given that $a = 10$ and $b = 9$ are the longer sides.

Therefore, $c = 5 + \sqrt{6}$ and $5 - \sqrt{6}$ both are possible.

$$5. \text{ We have } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0$$

The equation which is quadratic in ' c '.

$$\therefore c_1 + c_2 = 2b \cos A \text{ and } c_1 c_2 = b^2 - a^2 \quad \dots (i)$$

$$\begin{aligned} \therefore c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A &= (c_1 + c_2)^2 - 2c_1 c_2 - 2c_1 c_2 \cos 2A \quad [\text{using (i)}] \\ &= (c_1 + c_2)^2 - 2c_1 c_2 (1 + \cos 2A) \\ &= 4b^2 \cos^2 A - 2(b^2 - a^2) 2 \cos^2 A \\ &= 4a^2 \cos^2 A \end{aligned}$$

6. We have,

$$\cos A = \frac{c^2 + b^2 - a^2}{2bc}$$

$$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0$$

It is given that c_1 and c_2 are the roots of this equation.

$$\text{Therefore } c_1 + c_2 = 2b \cos A \text{ and } c_1 c_2 = b^2 - a^2.$$

$$\Rightarrow 2R (\sin C_1 + \sin C_2) = 4R \sin B \cos A$$

$$\Rightarrow \sin C_1 + \sin C_2 = 2 \sin B \cos A$$

Now, sum of the areas of two triangles

$$= \frac{1}{2} ab \sin C_1 + \frac{1}{2} ab \sin C_2$$

$$= \frac{1}{2} ab (\sin C_1 + \sin C_2)$$

$$= \frac{1}{2} ab (2 \sin B \cos A)$$

$$= ab \sin B \cos A$$

$$= b(b \sin A) \cos A$$

$$(\because a \sin B = b \sin A)$$

$$= \frac{1}{2} b^2 \sin 2A.$$

Exercise 5.7

1. Distance of circumcenter from to side BC is $R \cos A = f$

$$\text{Similarly, } g = R \cos B, h = R \cos C$$

$$\begin{aligned} \Rightarrow \frac{a}{f} + \frac{b}{g} + \frac{c}{h} &= \frac{2R \sin A}{R \cos A} + \frac{2R \sin B}{R \cos B} + \frac{2R \sin C}{R \cos C} \\ &= 2(\tan A + \tan B + \tan C) \end{aligned}$$

$$\text{Also, } \frac{a}{f} \frac{b}{g} \frac{c}{h} = 8 \tan A \tan B \tan C$$

But in triangle, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Thus,

$$\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \frac{1}{4} \frac{abc}{fgh}$$

2. Chord AC subtends the same angle at point B and D .

$$\therefore \angle ADC = B$$

So, in right angled triangle ACD ,

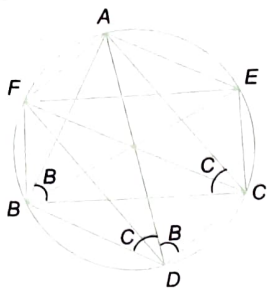
$$CD = AC \cot B$$

$$\therefore CD = b \cot B$$

Similarly from $\triangle ABD$,

$$BD = c \cot C$$

(as $\angle ADB = C$)



$\therefore$ Area of $\triangle BDC$,

$$\begin{aligned}\Delta_{BDC} &= \frac{1}{2} (b \cot B)(c \cot C) \sin(B+C) \\ &= \frac{bc}{2} \sin A \cot C \cot B \\ &= \frac{abc}{4R} \cot B \cot C\end{aligned}$$

$$\text{Similarly } \Delta_{AEC} = \frac{abc}{4R} \cot A \cot C$$

$$\text{and } \Delta_{AFB} = \frac{abc}{4R} \cot A \cot B$$

$\therefore$ Area of hexagon $AFBDC E$

$$\begin{aligned}&= \Delta_{BDC} + \Delta_{AFC} + \Delta_{AFB} + \Delta_{ABC} \\ &= \frac{abc}{4R} [\cot A \cot B + \cot B \cot C + \cot C \cot A] + \Delta \\ &= \Delta + \Delta \\ &\text{(as in } \triangle ABC, \cot A \cot B + \cot B \cot C + \cot C \cot A = 1) \\ &= 2\Delta\end{aligned}$$

3. We have $a : b : c = 3 : 5 : 7$.

Distance of circumcentre from sides are $R|\cos A|$, $R|\cos B|$ and $R|\cos C|$.

So, $R|\cos C|$ is minimum.

$$\cos C = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = \frac{9 + 25 - 49}{30} = \frac{-1}{2}$$

So, required minimum distance is $R/2$.

4. Given that $R = 4$ cm.

Sum of perpendicular distances from circumcentre to the sides of triangle,

$$\begin{aligned}S &= R(\cos A + \cos B + \cos C) \\ &= 4(\cos A + \cos B + \cos C) \leq 4(3/2)\end{aligned}$$

Thus, S cannot exceed 6.

Exercise 5.8

1. Distance between circumcenter (O) and incenter (I) is $\sqrt{R^2 - 2rR}$.

If incircle of triangle ABC passes through its circumcenter,

$$\sqrt{R^2 - 2rR} = r$$

$$\Rightarrow \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right) - 1 = 0$$

$$\Rightarrow \frac{r}{R} = \frac{-2 \pm \sqrt{4+4}}{2}$$

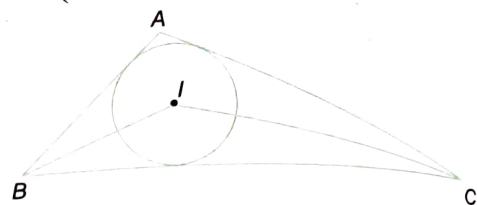
$$\Rightarrow \frac{r}{R} = \sqrt{2} - 1$$

$$\left(\text{as } \frac{r}{R} > 0\right)$$

$$\Rightarrow 1 + \frac{r}{R} = \sqrt{2}$$

$$\Rightarrow 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \sqrt{2} - 1$$

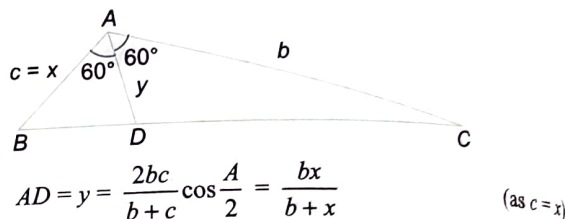
$$2. \angle BIC = \pi - \left(\frac{B+C}{2}\right) = \frac{\pi}{2} + \frac{A}{2}$$



Using the sine rule,

$$\begin{aligned}\text{Diameter of required circle} &= \frac{BC}{\sin \angle BIC} = \frac{a}{\cos \frac{A}{2}} = 20 \\ \therefore \text{Radius} &= 10\end{aligned}$$

3.

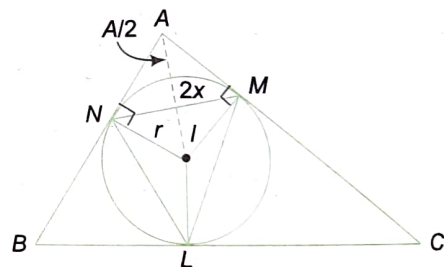


$$\text{But } bx = 1 \text{ or } b = \frac{1}{x}$$

$$\therefore y = \frac{x}{1+x^2} = \frac{1}{x + \frac{1}{x}}$$

Thus, $y_{\max} = \frac{1}{2}$, since the minimum value of the denominator is 2 if $x > 0$.

4. In the given figure, $ANIM$ is a cyclic quadrilateral.



Also, AI is the diameter of circumcircle MNI .

$\therefore AI = 2x$. Then

$$\operatorname{cosec} \frac{A}{2} = \frac{2x}{r}$$

$$\Rightarrow x = \frac{r}{2 \sin \frac{A}{2}}, y = \frac{r}{2 \sin \frac{B}{2}}, z = \frac{r}{2 \sin \frac{C}{2}}$$

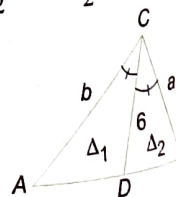
$$\Rightarrow xyz = \frac{r^3}{8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{r^3}{2 \frac{r}{R}} = \frac{r^2 R}{2}$$

5. $\Delta = \Delta_1 + \Delta_2$

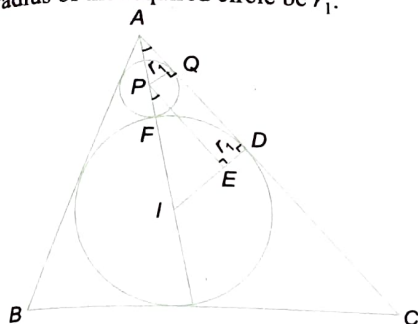
$$\text{or } \frac{1}{2} ab \sin C = \frac{1}{2} 6b \sin \frac{C}{2} + \frac{1}{2} 6a \sin \frac{C}{2}$$

$$\text{or } ab \sin \frac{C}{2} \cos \frac{C}{2} = \frac{1}{2} 6b \sin \frac{C}{2} + \frac{1}{2} 6a \sin \frac{C}{2}$$

$$\text{or } \frac{1}{a} + \frac{1}{b} = \frac{1}{9}$$



6. Let the radius of the required circle be r_1 .



As shown in the figure, angle between the direct common tangents AB and AC is 60° .

$$\text{In } \triangle PEI, \sin \frac{\pi}{6} = \frac{IE}{PI} = \frac{ID - ED}{PF + FI} = \frac{r - r_1}{r + r_1}$$

$$\therefore \frac{6 - r_1}{6 + r_1} = \frac{1}{2}$$

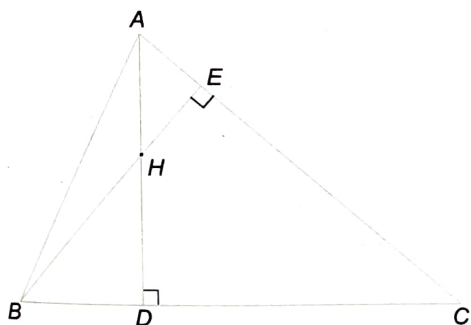
$$\Rightarrow r_1 = 2$$

7. For triangle ABC , we have

$$\begin{aligned} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} \\ &= \frac{\Delta}{s^2} = \frac{r}{s} \leq \frac{R}{2s} \end{aligned}$$

Exercise 5.9

1.



In figure in $\triangle ADB$, $BD = c \cos B$ (projection of AB on BC)

In $\triangle ADC$, $CD = b \cos C$ (projection of AC on BC)

$$\therefore \frac{BD}{CD} = \frac{c \cos B}{b \cos C} = \frac{2R \sin C \cos B}{2R \sin B \cos C} = \frac{\tan C}{\tan B}$$

$$\text{Also, } \frac{AH}{HD} = \frac{2R \cos A}{2R \cos B \cos C}$$

$$= \frac{\sin A}{\tan A \cos B \cos C}$$

$$= \frac{\sin(B+C)}{\tan A \cos B \cos C}$$

$$= \frac{\sin B \cos C + \sin C \cos B}{\tan A \cos B \cos C}$$

$$= \frac{\tan B + \tan C}{\tan A}$$

2. Distance of vertex A from orthocenter H given by is

$$AH = 2R \cos A$$

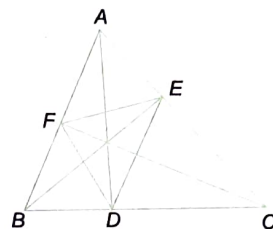
We have,

$$R = \frac{a}{2 \sin A} = \frac{2 + \sqrt{5}}{2 \sin 30^\circ} = \frac{2 + \sqrt{5}}{2 \times \frac{1}{2}} = (2 + \sqrt{5})$$

$$\therefore AH = 2(2 + \sqrt{5}) \cos 30^\circ$$

$$= (2 + \sqrt{5}) \sqrt{3}$$

3. AD , BE and CF are altitudes.



$$DE = c \cos C, EF = a \cos A, FD = b \cos B$$

$$\text{Given that } 4R = a \cos A + b \cos B + c \cos C$$

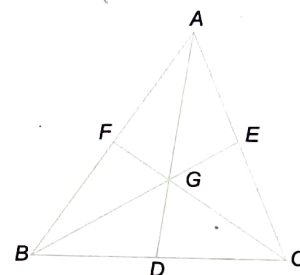
$$\Rightarrow 4R = R(\sin 2A + \sin 2B + \sin 2C)$$

$$\Rightarrow \sin A \sin B \sin C = 1$$

This is possible only when $\angle A = \angle B = \angle C = \pi/2$

So, triangle is not possible.

4. Points A , F , G and E are concyclic,



$$\Rightarrow BG \cdot BE = BF \cdot BA$$

$$\Rightarrow \frac{2}{3} (BE)^2 = \frac{1}{2} c^2$$

$$\Rightarrow \frac{2}{3} \times \frac{1}{4} (2a^2 + 2c^2 - b^2) = \frac{1}{2} c^2$$

$$\Rightarrow 2a^2 = b^2 + c^2$$

(Using Apollonius Theorem)

$$\begin{aligned} 5. \frac{FE}{a} + \frac{FD}{b} + \frac{DE}{c} &= \frac{a \cos A}{a} + \frac{b \cos B}{b} + \frac{c \cos C}{c} \\ &= \cos A + \cos B + \cos C \\ &\leq \frac{3}{2} \end{aligned}$$

Exercise 5.10

1. In $\triangle ABC$, we have $r_1 < r_2 < r_3$. Thus,

$$\frac{1}{r_1} > \frac{1}{r_2} > \frac{1}{r_3}$$

$$\Rightarrow \frac{s-a}{\Delta} > \frac{s-b}{\Delta} > \frac{s-c}{\Delta}$$

$$\Rightarrow s-a > s-b > s-c$$

$$\Rightarrow -a > -b > -c$$

$$\text{or } a < b < c$$

2. r_1, r_2, r_3 are in H.P.

$$\Rightarrow 1/r_1, 1/r_2, 1/r_3 \text{ are in A.P.}$$

$$\Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta} \text{ are in A.P.}$$

$$\Rightarrow s-a, s-b, s-c \text{ are in A.P.}$$

$$\Rightarrow a, b, c \text{ are in A.P.}$$

3. Given, $(a-b)(s-c) = (b-c)(s-a)$

$$\therefore \frac{a-b}{s-a} = \frac{b-c}{s-c}$$

$$\Rightarrow \frac{(s-b)-(s-a)}{(s-a)(s-b)} = \frac{(s-c)-(s-b)}{(s-b)(s-c)}$$

$$\Rightarrow \frac{1}{s-a} - \frac{1}{s-b} = \frac{1}{s-b} - \frac{1}{s-c}$$

$$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s-b} = \frac{\Delta}{s-b} - \frac{\Delta}{s-c}$$

$$\Rightarrow r_1 - r_2 = r_2 - r_3$$

Hence, r_1, r_2, r_3 are in A.P.

$$4. 2R + r - r_1 = 2R + (s-a) \tan \frac{A}{2} - s \tan \frac{A}{2}$$

$$= 2R - a \tan \frac{A}{2}$$

$$= 2R - 2R \sin A \tan \frac{A}{2}$$

$$= 2R \left(1 - 2 \sin^2 \frac{A}{2} \right)$$

$$= 2R \cos A$$

$$5. \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

$$\text{Also, } \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

$$\left\{ \because \Delta = \frac{1}{2} a \cdot p_1 = \frac{1}{2} b \cdot p_2 = \frac{1}{2} c \cdot p_3 \right\}$$

$$\text{Hence, } \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

6. We have already proved that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

$$\Rightarrow r_2 r_3 + r_3 r_1 + r_1 r_2 = \frac{r_1 r_2 r_3}{r} = \frac{\Delta^3 s}{(s-a)(s-b)(s-c) \Delta} = s^2$$

7. We have already proved that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$ Now using A.M. $\geq$ H.M., we get

$$\frac{r_1 + r_2 + r_3}{3} \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = 3r$$

$$\Rightarrow \frac{r_1 + r_2 + r_3}{r} \geq 9$$

$$8. \frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{\frac{\Delta}{s-a} - \frac{\Delta}{s}}{a} + \frac{\frac{\Delta}{s-b} - \frac{\Delta}{s}}{b}$$

$$= \frac{\Delta(s-s+a)}{a(s-a)s} + \frac{\Delta(s-s+b)}{s(s-b)b}$$

$$= \frac{\Delta}{s(s-a)} + \frac{\Delta}{s(s-b)}$$

$$= \frac{\Delta}{s} \left(\frac{s-b+s-a}{(s-a)(s-b)} \right)$$

$$= \frac{\Delta}{s} \frac{c(s-c)}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta c(s-c)}{\Delta^2}$$

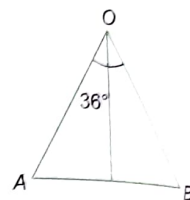
$$= \frac{c}{r_3}$$

Exercise 5.11

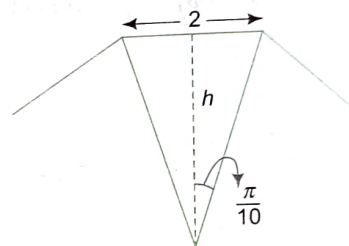
$$1. \Delta_1 = \frac{5R_1^2}{2} \sin 72^\circ$$

$$\Delta_2 = \frac{5R_2^2}{2} \sin 72^\circ$$

$$\Rightarrow \frac{\Delta_1}{\Delta_2} = \frac{25}{4}$$



$$2. \text{ In the figure, } h = \frac{1}{\tan \frac{\pi}{10}}$$

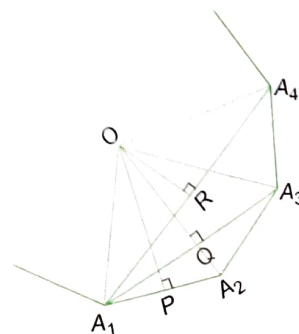


$$\text{Area of polygon} = 10 \left(\frac{1}{2} \cdot 2 \cdot \frac{1}{\tan \frac{\pi}{10}} \right) = \frac{10}{\tan \frac{\pi}{10}}$$

Now, from point A inside polygon, draw perpendiculars from it to the sides of polygon.

$$\text{Then the area of polygon} = \sum_{i=1}^n \frac{1}{2} \cdot 2p_i = \frac{10}{\tan \frac{\pi}{10}}$$

$$\therefore p_1 + p_2 + \dots + p_{10} = \frac{10}{\tan \frac{\pi}{10}}$$

3. Let a be the side of n sided regular polygon $A_1 A_2 A_3 A_4 \dots A_n$ 

$$\therefore \text{Angle subtended by each side at centre} = \frac{2\pi}{n}$$

Also $OA_1 = OA_2 = OA_3 = \dots = OA_n = r$ (Say)

In ΔA_1OA_2 , using cosine rule

$$\begin{aligned} A_1A_2^2 &= r^2 + r^2 - 2r^2 \cos \frac{2\pi}{n} \\ &= 2r^2 \left(1 - \cos \frac{2\pi}{n} \right) \\ &= 4r^2 \sin^2 \frac{\pi}{n} \end{aligned}$$

$$\therefore A_1A_2 = 2r \sin \frac{\pi}{n}$$

Similarly in ΔA_1OA_3 , $A_1A_3 = 2r \sin \frac{2\pi}{n}$

and in ΔA_1OA_4 , $A_1A_4 = 2r \sin \frac{3\pi}{n}$

$$\text{But given that } \frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

$$\Rightarrow \frac{1}{2r \sin \frac{\pi}{n}} = \frac{1}{2r \sin \frac{2\pi}{n}} + \frac{1}{2r \sin \frac{3\pi}{n}}$$

$$\Rightarrow 2 \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} = 2 \sin \frac{\pi}{n} \sin \frac{3\pi}{n} + 2 \sin \frac{\pi}{n} \sin \frac{2\pi}{n}$$

$$\Rightarrow \cos \frac{\pi}{n} - \cos \frac{5\pi}{n} = \cos \frac{2\pi}{n} - \cos \frac{4\pi}{n} + \cos \frac{\pi}{n} - \cos \frac{3\pi}{n}$$

$$\Rightarrow \cos \frac{5\pi}{n} + \cos \frac{2\pi}{n} = \cos \frac{4\pi}{n} + \cos \frac{3\pi}{n}$$

$$\Rightarrow 2 \cos \frac{7\pi}{2n} \cos \frac{3\pi}{2n} = 2 \cos \frac{7\pi}{2n} \cos \frac{\pi}{2n}$$

$$\Rightarrow \cos \frac{7\pi}{2n} \left(\cos \frac{3\pi}{2n} - \cos \frac{\pi}{2n} \right) = 0$$

$$\Rightarrow 2 \cos \frac{7\pi}{2n} \sin \frac{\pi}{n} \sin \frac{\pi}{2n} = 0$$

$$\Rightarrow \frac{7\pi}{2n} = (2k+1) \frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{n} = k\pi \quad \text{or} \quad \frac{\pi}{2n} = k\pi, k \in \mathbb{Z}$$

$$\Rightarrow n = \frac{7}{2k+1} \quad \text{or} \quad n = \frac{1}{k} \quad (\text{not possible})$$

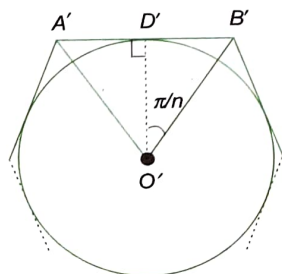
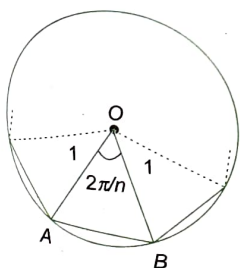
$$\text{or } n = \frac{1}{2k} \quad (\text{not possible})$$

$$\therefore n = 7 \text{ for } k = 0$$

4. Let OAB be one triangle out of n of a n -sided polygon inscribed in a circle of radius 1. Then,

$$\angle AOB = \frac{2\pi}{n}$$

$$OA = OB = 1$$



$$\begin{aligned} \therefore \text{Area } (\Delta AOB) &= \frac{1}{2} \times 1 \times 1 \times \sin \left(\frac{2\pi}{n} \right) \\ &= \frac{1}{2} \sin \left(\frac{2\pi}{n} \right) \end{aligned}$$

$$\text{Area of } n\text{-sided polygon} = I_n = \frac{n}{2} \sin \frac{2\pi}{n} \quad (i)$$

Similarly, let $O'A'B'$ be one of the triangles out of n of n -sided polygon circumscribing the circle of unit radius.

$$\text{Then in } \Delta O'B'A', \cos \frac{\pi}{n} = \frac{1}{O'B'}$$

$$\text{or } O'B' = \sec \frac{\pi}{n}$$

$$\begin{aligned} \text{Area } (\Delta O'B'A') &= \frac{1}{2} (O'B')^2 \sin \left(\frac{2\pi}{n} \right) \\ &= \frac{1}{2} \sec^2 \left(\frac{\pi}{n} \right) \sin \left(\frac{2\pi}{n} \right) \end{aligned}$$

Therefore, the area of n -sided polygon is given by

$$(O_n) = \frac{n}{2} \sec^2 \frac{\pi}{n} \sin \frac{2\pi}{n} \quad (ii)$$

From Eqs. (i) and (ii), we get

$$\frac{I_n}{O_n} = \frac{\frac{n}{2} \sin \frac{2\pi}{n}}{\frac{n}{2} \sec^2 \frac{\pi}{n} \sin \frac{2\pi}{n}} = \frac{1}{\sec^2 \frac{\pi}{n}}$$

$$\begin{aligned} \text{or } I_n &= \left(\cos^2 \frac{\pi}{n} \right) O_n \\ &= \frac{O_n}{2} \left[1 + \cos \left(\frac{2\pi}{n} \right) \right] \\ &= \frac{O_n}{2} \left[1 + \sqrt{1 - \sin^2 \frac{2\pi}{n}} \right] \\ &= \frac{O_n}{2} \left[1 - \sqrt{1 - \left(\frac{2I_n}{n} \right)^2} \right] \end{aligned}$$

Exercises

Single Correct Answer Type

$$\begin{aligned} 1. (4) \quad & \frac{\sin A (a - b \cos C)}{\sin C (c - b \cos A)} \\ &= \frac{\sin A (b \cos C + c \cos B - b \cos C)}{\sin C (a \cos B + b \cos A - b \cos A)} \\ &= \frac{\sin A (c \cos B)}{\sin C (a \cos B)} \\ &= 1 \quad (\text{as } c \sin A = a \sin C) \end{aligned}$$

$$\begin{aligned} 2. (2) \quad \text{L.H.S} &= \frac{1 + \cos A}{a} + \frac{1 + \cos B}{b} + \frac{1 + \cos C}{c} \\ &= \frac{2 \cos^2 \frac{A}{2}}{2R \sin A} + \frac{2 \cos^2 \frac{B}{2}}{2R \sin B} + \frac{2 \cos^2 \frac{C}{2}}{2R \sin C} \\ &= \frac{1}{2R} \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right) \end{aligned}$$

$$\begin{aligned}\text{Now, } \frac{1+\cos A}{a} \cdot \frac{1+\cos B}{b} \cdot \frac{1+\cos C}{c} \\&= \frac{\cos^2 \frac{A}{2}}{R \sin A} \cdot \frac{\cos^2 \frac{B}{2}}{R \sin B} \cdot \frac{\cos^2 \frac{C}{2}}{R \sin C} \\&= \frac{1}{8R^3} \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}\end{aligned}$$

3. (2) We know that $A + B + C = 180^\circ$

$$\text{or } A + C - B = 180^\circ - 2B$$

$$\begin{aligned}\text{Now, } 2ac \sin \left[\frac{1}{2}(A - B + C) \right] \\&= 2ac \sin(90^\circ - B) = 2ac \cos B \\&= \frac{2ac(a^2 + c^2 - b^2)}{2ac} = a^2 + c^2 - b^2\end{aligned}$$

4. (1) Given that $4A + A + A = 180^\circ$

$$\text{or } A = 30^\circ$$

Angles are $120^\circ, 30^\circ, 30^\circ$

$$\Rightarrow \frac{\sin 120^\circ}{a} = \frac{\sin 30^\circ}{b} = \frac{\sin 30^\circ}{c} = 2R \text{ (say)}$$

$$\Rightarrow \frac{a}{a+b+c} = \frac{\sin 120^\circ}{\sin 120^\circ + \sin 30^\circ + \sin 30^\circ} = \frac{\sqrt{3}}{2+\sqrt{3}}$$

5. (4) By sine law in $\triangle ABC$, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(\pi - A - B)} = 2R$$

$$\text{or } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(A+B)} = 2R$$

(1) If we know $a, \sin A, \sin B$, we can find b, c , and the value of angles A, B , and C .

(2) With a, b, c , we can find $\angle A, \angle B, \angle C$ using the cosine law.

(3) $a, \sin B, R$ are given, so $\sin A, b$ and hence $\sin(A+B)$ and then C can be found.

(4) If we know $a, \sin A, R$, then we can get the ratio $\frac{b}{\sin B}$ or $\frac{c}{\sin(A+B)}$ only. We cannot determine the values of $b, c, \sin B, \sin C$ separately. Therefore, the triangle cannot be determined uniquely in this case.

6. (4) Sides are in the ratio $1 : \sqrt{3} : 2$

$$\text{Let } a = k, b = \sqrt{3}k, \text{ and } c = 2k$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\sqrt{3}}{2} \Rightarrow A = \frac{\pi}{6}$$

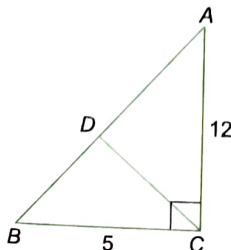
$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{1}{2} \Rightarrow B = \frac{\pi}{3}$$

$$\Rightarrow C = \pi - A - B = \frac{\pi}{2}$$

$$\Rightarrow A:B:C = 1:2:3$$

7. (3) In $\triangle CDB$, $\frac{CD}{\sin B} = \frac{BC}{\sin(135^\circ - B)}$

$$\begin{aligned}\Rightarrow CD &= \frac{BC \sin B}{\sin(135^\circ - B)} \\&= \frac{5\sqrt{2}}{1 + \cot B} = \frac{5\sqrt{2}}{1 + \frac{5}{12}}\end{aligned}$$



$$\therefore CD = \frac{60\sqrt{2}}{17}$$

$$\text{Also from same triangle, } \frac{CD}{\sin B} = \frac{BD}{\sin 45^\circ}$$

$$\Rightarrow BD = \frac{65}{17}$$

$$\Rightarrow AD = 13 - \frac{65}{17} = \frac{156}{17}$$

8. (3) $(a+b+c)(b+c-a) = kbc$

$$\text{or } (b+c)^2 - a^2 = kbc$$

$$\text{or } b^2 + c^2 - a^2 = (k-2)bc$$

$$\text{or } 2bc \cos A = (k-2)bc$$

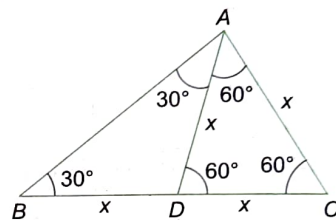
$$\text{or } \cos A = \frac{k-2}{2}$$

Now, A being the angle of a triangle, we have

$$-1 < \cos A < 1 \quad \text{or} \quad -2 < k-2 < 2$$

$$\text{or } 0 < k < 4$$

9. (2)



$$\text{In } \triangle ABD, \cos 120^\circ = \frac{x^2 + x^2 - AB^2}{2x^2}$$

$$\Rightarrow \frac{2x^2 - AB^2}{2x^2} = \frac{-1}{2}$$

$$\Rightarrow 3x^2 = AB^2$$

$$\Rightarrow AB = x\sqrt{3}$$

$$\begin{aligned}\Rightarrow a^2 : b^2 : c^2 &= (2x)^2 : x^2 : (x\sqrt{3})^2 \\&= 4x^2 : x^2 : 3x^2 = 4 : 1 : 3\end{aligned}$$

10. (1) Given $c \sin B \geq a$

$$\Rightarrow \sin C \sin B \geq \sin A$$

$$\text{and } a \sin C \geq b$$

$$\Rightarrow \sin C \sin A \geq \sin B$$

$$\Rightarrow \sin C \cdot \sin A \geq \sin B \geq \frac{\sin A}{\sin C}$$

$$\therefore \sin^2 C \geq 1$$

$$\Rightarrow \sin C = 1$$

$$\Rightarrow \angle C \text{ is } \frac{\pi}{2}$$

11. (2) $\frac{\sin A}{c \sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$

$$\text{or } \frac{a}{bc} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$$

$$\text{or } \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac}$$

$$\text{or } \frac{b \sin B + c \sin C}{bc} = \frac{c^2 + b^2}{abc}$$

$$\text{or } a = \frac{b^2 + c^2}{b \sin B + c \sin C}$$

$$= \frac{b(2R \sin B) + c(2R \sin C)}{b \sin B + c \sin C}$$

$$= 2R$$

$$\Rightarrow \angle A = \frac{\pi}{2}$$

$$(3) \quad 2b = a + c \Rightarrow 2 \sin B = \sin A + \sin C$$

$$\text{or } 4 \sin \frac{B}{2} \cos \frac{B}{2} = 2 \sin \left(\frac{A+C}{2} \right) \cos \left(\frac{A-C}{2} \right)$$

$$\Rightarrow \sin \left(\frac{B}{2} \right) = \frac{1}{2} \cos(A-C) \leq \frac{1}{2} \Rightarrow B \leq 60^\circ$$

$$(4) \quad \Delta = \frac{1}{2} a \times AD$$

$$\Rightarrow AD = \frac{2\Delta}{a} = \frac{abc}{b^2 - c^2} \quad (\text{given})$$

$$\Rightarrow \frac{2\Delta}{a} = \frac{4R \cdot \Delta}{b^2 - c^2}$$

$$\Rightarrow 2Ra = b^2 - c^2$$

$$\Rightarrow \sin A = \sin^2 B - \sin^2 C \quad (\text{Using Sine rule})$$

$$= \sin(B+C) \sin(B-C)$$

$$= \sin A \sin(B-C)$$

$$\Rightarrow \sin(B-C) = 1$$

$$\Rightarrow \angle B - \angle C = \frac{\pi}{2}$$

$$\Rightarrow \angle B = \frac{\pi}{2} + \angle C = 90^\circ + 23^\circ = 113^\circ$$

$$(4) \quad \text{Let the sides of the triangle be } a, ar, ar^2.$$

$$\because ar^2 \text{ is the longest side } (r > 1)$$

$$\therefore a + ar > ar^2$$

$$\text{So, } \cos C = \frac{a^2 + a^2 r^2 - a^2 r^4}{2a^2 r} = \frac{1 + r^2 - r^4}{2r} < \frac{1}{2}$$

$$\Rightarrow C > \frac{\pi}{3}$$

$$\text{Also, } \cos B = \frac{a^2 + a^2 r^4 - a^2 r^2}{2a^2 r^2}$$

$$= \frac{1 + r^4 - r^2}{2r^2}$$

$$= \frac{1}{2} \left[r^2 + \frac{1}{r^2} - 1 \right]$$

$$= \frac{1}{2} \left[\left(r - \frac{1}{r} \right)^2 + 1 \right] > 1/2$$

$$\Rightarrow B < \pi/3$$

$$\text{Also, } a < ar < ar^2.$$

$$\Rightarrow A < B < C$$

$$\Rightarrow A < B < \frac{\pi}{3} < C$$

$$15. (2) \quad b^2 \sin 2C + c^2 \sin 2B = 2bc$$

$$\Rightarrow b^2 (2 \sin C \cos C) + c^2 (2 \sin B \cos B) = 2bc$$

$$\Rightarrow b^2 (c \cos C) + c^2 (b \cos B) = 2Rbc$$

$$\Rightarrow b \cos C + c \cos B = 2R$$

$$\Rightarrow a = 2R$$

$$\Rightarrow A = 90^\circ$$

$$\therefore a^2 = b^2 + c^2 = 400 + 441 = 841$$

$$\Rightarrow a = 29$$

$$\text{Now, } r = \frac{2\Delta}{2s} = \frac{20 \times 21}{20 + 21 + 29} = \frac{20 \times 21}{70} = 6$$

16. (2) Using cosine rule, we get

$$x^2 = (x+1)^2 + b^2 - 2(x+1)b \cos \frac{\pi}{3}$$

$$\Rightarrow 0 = 2x + 1 + b^2 - (x+1)b$$

$$\Rightarrow b^2 - (x+1)b + 2x + 1 = 0$$

Since b is real, we have

$$\Rightarrow (x+1)^2 - 4(2x+1) \geq 0$$

$$\Rightarrow x^2 - 6x - 3 \geq 0$$

$$\Rightarrow x \geq 3 + \sqrt{12}$$

The least integral value of x is 7.

17. (3) $a = 2b$ and $A - B = 60^\circ$

$$\text{We know that } \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

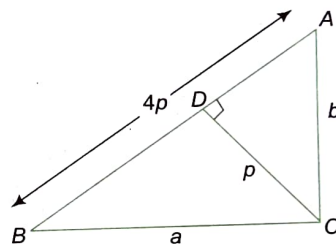
$$\text{or } \tan 30^\circ = \frac{1}{3} \cot \frac{C}{2}$$

$$\text{or } \tan \frac{C}{2} = \frac{1}{\sqrt{3}} \text{ or } C = 60^\circ$$

$$\text{Hence, } A + B = 120^\circ \text{ or } 2A = 180^\circ$$

$$\Rightarrow A = 90^\circ, B = 30^\circ, C = 60^\circ$$

18. (1)



$$\Delta = \frac{1}{2} ab = \frac{1}{2} p (4p) \Rightarrow ab = 4p^2$$

$$\text{Also, } a^2 + b^2 = c^2 = 16p^2$$

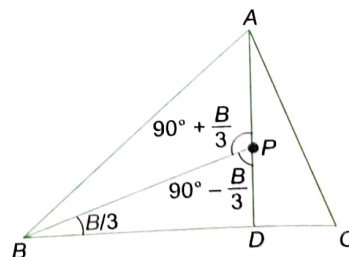
$$\therefore (a-b)^2 = a^2 + b^2 - 2ab = 8p^2$$

$$\text{Also, } (a+b)^2 = a^2 + b^2 + 2ab = 24p^2$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{\sqrt{3}} \times 1$$

$$\text{or } \frac{A-B}{2} = 30^\circ \text{ or } A - B = 60^\circ$$

19. (3)



$$\angle BPA = 90^\circ + \left(\frac{B}{3} \right), \angle ABP = \frac{2B}{3}$$

$$\text{In } \Delta ABP, \frac{AP}{\sin(2B/3)} = \frac{c}{\sin[90^\circ + (B/3)]} = \frac{c}{\cos(B/3)}$$

[by the sine rule]

$$\begin{aligned}\text{or } AP &= \frac{c \sin(2B/3)}{\cos(B/3)} = \frac{2c \sin(B/3) \cos(B/3)}{\cos(B/3)} \\ &= 2c \sin(B/3)\end{aligned}$$

$$\begin{aligned}20. (1) \quad & a \cos(B-C) + b \cos(C-A) + c \cos(A-B) \\ &= 2R \sin A \cos(B-C) + 2R \sin B \cos(C-A) + 2R \sin C \cos(A-B) \\ &= 2R \sin(B+C) \cos(B-C) + 2R \sin(A+C) \cos(C-A) \\ &\quad + 2R \sin(A+B) \cos(A-B) \\ &= R[\sin 2B + \sin 2C + \sin 2C + \sin 2A + \sin 2A + \sin 2B] \\ &= 2R(\sin 2A + \sin 2B + \sin 2C) \\ &= 8R \sin A \sin B \sin C \\ &= 8R \frac{a}{2R} \frac{b}{2R} \frac{c}{2R} = \frac{abc}{R^2}\end{aligned}$$

21. (1) We have

$$\begin{aligned}8R^2 &= a^2 + b^2 + c^2 \\ &= 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \\ \text{or } \sin^2 A + \sin^2 B + \sin^2 C &= 2 \\ \text{or } 1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C &= 2 \\ \text{or } (\cos^2 A - \sin^2 C) + \cos^2 B &= 0 \\ \text{or } \cos(A+C) \cos(A-C) + \cos^2 B &= 0 \\ \text{or } -\cos B [\cos(A-C) - \cos B] &= 0 \\ \text{or } -\cos B [\cos(A-C) + \cos(A+C)] &= 0 \\ \text{or } -2 \cos A \cos B \cos C &= 0 \\ \text{or } \cos A = 0 \text{ or } \cos B = 0 \text{ or } \cos C &= 0 \\ \text{or } A = \frac{\pi}{2} \text{ or } B = \frac{\pi}{2} \text{ or } C = \frac{\pi}{2}\end{aligned}$$

22. (2) Let $OP = x$ Circumradius = R

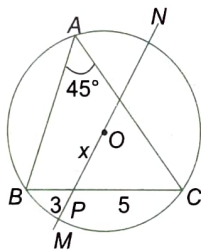
$$\therefore \frac{BC}{\sin 45^\circ} = 2R$$

$$\text{or } R = 4\sqrt{2}$$

$$\text{Now } PB \cdot PC = PM \cdot PN$$

$$\text{or } 15 = (R-x)(R+x)$$

$$\text{or } x = \sqrt{17}$$

23. (3) In triangle ABC ,

$$\begin{aligned}\frac{a^2 + b^2 + c^2}{R^2} &= 4(\sin^2 A + \sin^2 B + \sin^2 C) \\ &= 8(1 + \cos A \cos B \cos C) \leq 8\left(1 + \frac{1}{8}\right) \leq 9\end{aligned}$$

$$24. (3) \quad R(b+c) = a\sqrt{bc} = 2R \sin A \sqrt{bc}$$

$$\therefore \sin A = \frac{b+c}{2\sqrt{bc}}$$

$$\text{Now } \sin A \leq 1$$

$$\Rightarrow \frac{b+c}{2\sqrt{bc}} \leq 1$$

$$\text{or } (\sqrt{b} - \sqrt{c})^2 \leq 0$$

$$\text{or } b = c$$

$$\Rightarrow \sin A = 1$$

$$\Rightarrow A = 90^\circ \text{ and } b = c$$

Hence, the triangle is right isosceles.

$$25. (1) \quad \frac{\cot A}{\cot B + \cot C} = \frac{\frac{R(b^2 + c^2 - a^2)}{abc}}{\frac{R(a^2 + c^2 - b^2)}{abc} + \frac{R(a^2 + b^2 - c^2)}{abc}}$$

$$\begin{aligned}&= \frac{b^2 + c^2 - a^2}{2a^2} \\ &= \frac{2a^2 - a^2}{2a^2} \\ &= \frac{1}{2}\end{aligned}$$

$$26. (3) \quad \text{Here, } \sin \theta - \cos \theta = \frac{b}{a} \text{ and } \sin \theta \cos \theta = \frac{c}{a}$$

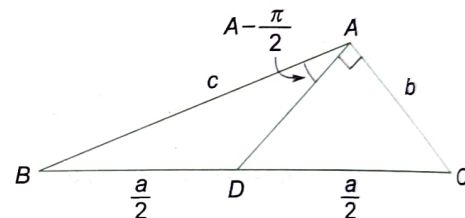
$$\text{or } 1 - 2 \sin \theta \cos \theta = \frac{b^2}{a^2}$$

$$\text{or } 1 - \frac{2c}{a} = \frac{b^2}{a^2}$$

$$\text{or } a^2 - b^2 = 2ac$$

$$\text{Hence, } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{2ac + c^2}{2ac} = 1 + \frac{c}{2a}$$

27. (1) From the right-angled $\triangle CAD$, we have

$$\cos C = \frac{b}{a/2} \Rightarrow \frac{2b}{a} = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\text{or } a^2 + b^2 - c^2 = 4b^2$$

$$\text{or } a^2 - c^2 = 3b^2$$

$$28. (1) \quad \cot A = \frac{\cos A}{\sin A} = \frac{\frac{b^2 + c^2 - a^2}{2bc}}{\frac{a}{2R}} = \left(\frac{b^2 + c^2 - a^2}{abc} \right) R$$

$$\therefore \cot A : \cot B : \cot C$$

$$= b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2$$

Clearly if $\cot A : \cot B : \cot C = l : m : n$,

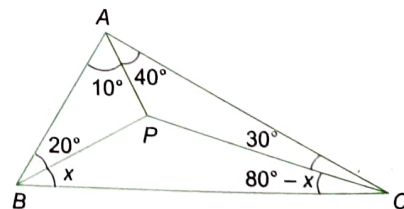
$$\text{then } a : b : c = \sqrt{m+n} : \sqrt{n+l} : \sqrt{l+m}$$

$$= \sqrt{19+6} : \sqrt{6+30} : \sqrt{30+19}$$

$$= 5 : 6 : 7$$

Therefore, a, b, c are in A.P.

29. (1)

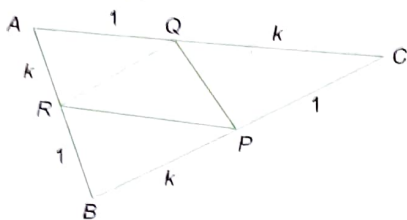
From $\triangle APB$, $\triangle PBC$ and $\triangle PCA$, using sine rule

$$\begin{aligned}&\frac{AP}{\sin \angle ABP} \cdot \frac{BP}{\sin \angle PCB} \cdot \frac{CP}{\sin \angle PAC} \\ &= \frac{AP}{\sin \angle PCA} \cdot \frac{CP}{\sin \angle PBC} \cdot \frac{BP}{\sin \angle PAB}\end{aligned}$$

$$\text{Also, } \angle BOD = \angle COD = \frac{2\pi}{3 \times 2} = \frac{\pi}{3}$$

$$\text{or } \tan \frac{\pi}{3} = \frac{BD}{OB} = \frac{l}{r} \Rightarrow l = r\sqrt{3} = d\sqrt{3}$$

39. (2) Area of triangle



$$\text{Area of } \triangle ARQ = \frac{b}{(k+1)} \times \frac{kc}{(k+1)} \times \frac{1}{2} \sin A = \frac{k\Delta}{(k+1)^2}$$

$$\text{Similarly, area } (\triangle BPR) = \text{Area } (\triangle PCQ) = \frac{k\Delta}{(k+1)^2}$$

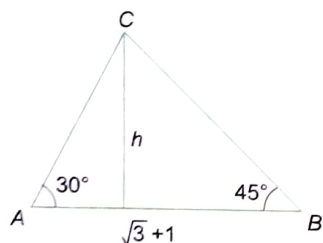
$$\text{Now, } \frac{\Delta_{PQR}}{\Delta_{ABC}} = \frac{1}{3}$$

$$\text{or } \frac{\Delta - \frac{3k}{(k+1)^2} \Delta}{\Delta} = \frac{1}{3}$$

$$\text{or } 2k^2 - 5k + 2 = 0$$

$$\text{or } k = \frac{1}{2}, 2$$

40. (1)



$$\angle C = 180^\circ - (30^\circ + 45^\circ) = 105^\circ$$

 By Sine Rule in $\triangle ABC$, we get

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 45^\circ} = \frac{\sqrt{3} + 1}{\sin 105^\circ}$$

$$\Rightarrow \frac{a}{1/2} = \frac{b}{1/\sqrt{2}} = \frac{\sqrt{3} + 1}{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \frac{a}{\sqrt{2}} = \frac{b}{2} = 1$$

$$\Rightarrow a = \sqrt{2}, b = 2$$

$$\Rightarrow \Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times \sqrt{2} \times 2 \times \sin 105^\circ$$

$$= \sqrt{2} \left(\frac{\sqrt{3} + 1}{2\sqrt{2}} \right)$$

$$= \frac{\sqrt{3} + 1}{2} \text{ sq. units.}$$

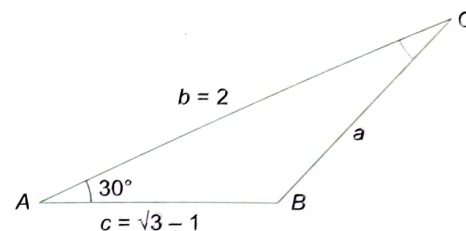
$$41. (1) \Delta = \frac{1}{2} (BC) h, \text{ where } h \text{ is the distance of vertex } A \text{ from side } BC$$

$$\Delta_{GBC} = \frac{\Delta}{3} = \frac{(BC)h}{6}, \text{ where } G \text{ is the centroid}$$

$$\Rightarrow h = \frac{2\Delta}{BC} = \text{constant}$$

Thus, distance of vertex A from the side is fixed. This, in turn, implies that the distance of centroid from side BC will be fixed, hence, locus of G will be a line parallel to BC .

42. (1)



$$\text{Using } \Delta = \frac{1}{2} bc \sin A, \text{ we get}$$

$$\frac{1}{2} \times 2 \times (\sqrt{3} - 1) \sin A = \frac{\sqrt{3} - 1}{2}$$

$$\therefore \sin A = \frac{1}{2} \text{ or } A = 30^\circ$$

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$

$$= \frac{3-\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}+1}{\sqrt{3}-1} = \sqrt{3}$$

$$\Rightarrow B - C = 120^\circ$$

$$\text{Also } B + C = 150^\circ \Rightarrow C = 15^\circ$$

$$43. (4) r = \frac{\Delta}{s} = 1 \Rightarrow s = 6$$

$$\therefore (s-a)(s-b)(s-c) = s$$

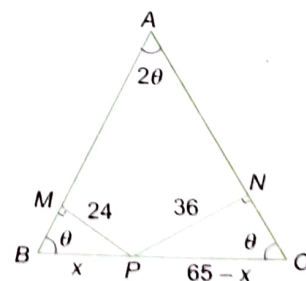
$$\Rightarrow s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc = s$$

$$\Rightarrow 6^3 - 12 \times 36 + (ab+bc+ca)(6) - 60 = 6$$

$$\Rightarrow ab+bc+ca = 47$$

$$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{47}{60} = 0.8 \text{ (nearly)}$$

44. (3)



$$\Delta = \frac{1}{2} b^2 \sin 2\theta = b^2 \sin \theta \cos \theta$$

$$\text{Now, } \frac{x}{24} = \frac{65-x}{36}$$

$$\text{or } 60x = (24)(65) \text{ or } x = 26$$

$$\therefore \sin \theta = \frac{12}{13} \text{ and } \cos \theta = \frac{5}{13}$$

$$(\because \triangle PMB \cong \triangle PNC)$$

$$\text{Now, } \frac{b}{\sin \theta} = \frac{65}{\sin 2\theta}$$

$$\text{or } b = \frac{65}{2 \cos \theta} = \frac{(65)(13)}{(2)(5)} = \frac{13^2}{2}$$

From Eq. (i), we get

$$\Delta = \frac{13^4}{4} \times \frac{12}{13} \times \frac{5}{13} = (169)(15) = 2535$$

Ex (2) We have

$$\Delta = \frac{\sqrt{3}}{4} a^2, s = \frac{3a}{2}$$

$$\therefore r = \frac{\Delta}{s} = \frac{a}{2\sqrt{3}}, R = \frac{abc}{4\Delta} = \frac{a^3}{\sqrt{3}a^2} = \frac{a}{\sqrt{3}}$$

$$\text{and } r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{3}/4 a^2}{a/2} = \frac{\sqrt{3}}{2} a$$

$$\text{Hence, } r : R : r_1 = \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}}{2} a = 1 : 2 : 3$$

$$\text{Ex (2) } \cos A + \cos B + \cos C = \frac{7}{4}$$

$$\Rightarrow 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{7}{4}$$

$$\text{or } 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{3}{4}$$

$$[\because r = 4R \sin(A/2) \sin(B/2) \sin(C/2)]$$

$$\text{or } \frac{r}{R} = \frac{3}{4} \quad \text{or} \quad \frac{R}{r} = \frac{4}{3}$$

Ex (2) Let a and b be the roots of $x^2 - 7x + 8 = 0$.

$$\text{Then } a + b = 7, ab = 8.$$

$$\text{Also, } C = 60^\circ$$

$$\Rightarrow c^2 = a^2 + b^2 - ab$$

$$\Rightarrow c^2 = (a+b)^2 - 3ab = 49 - 24 = 25$$

$$\Rightarrow c = 5$$

$$\therefore r \cdot R = \frac{abc}{2(a+b+c)} = \frac{8 \times 5}{2(7+5)} = \frac{5}{3}$$

Ex (1) $b = 2, c = \sqrt{3}, \angle A = 30^\circ$

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}$$

$$= \sqrt{4 + 3 - 2 \times 2 \times \sqrt{3} \times \frac{\sqrt{3}}{2}} = 1$$

$$\Rightarrow r = (s-a) \tan \frac{A}{2}$$

$$= \frac{b+c-a}{2} \tan \frac{A}{2}$$

$$= \frac{\sqrt{3}+1}{2} \tan 15^\circ$$

$$= \frac{\sqrt{3}+1}{2} \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2}$$

$$\text{Ex (4) } R = 8r = 8 \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

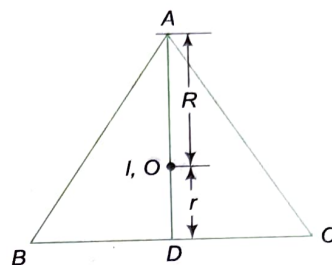
$$\therefore 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{1}{16}$$

$$\text{or } \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right) \sin \frac{C}{2} = \frac{1}{16}$$

$$\text{or } \sin \frac{C}{2} \left(\frac{1}{2} - \sin \frac{C}{2} \right) = \frac{1}{16} \Rightarrow \sin \frac{C}{2} = \frac{1}{4}$$

$$\text{or } \cos C = 1 - 2 \sin^2 \frac{C}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

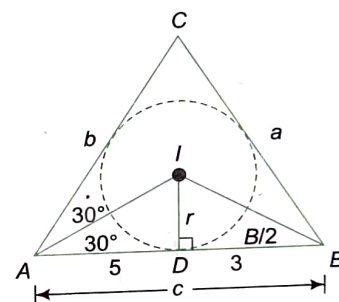
50. (3) In equilateral triangle, circumcenter (O) and incenter (I) coincide.



Also from the diagram,

$$R + r = h \quad \text{or} \quad \frac{R+r}{h} = 1$$

51. (3)



$$\tan 30^\circ = \frac{r}{5}$$

$$\text{or } r = \frac{5}{\sqrt{3}}$$

$$\text{We have } \tan \frac{B}{2} = \frac{r}{3} = \frac{5}{3\sqrt{3}}$$

...(i)

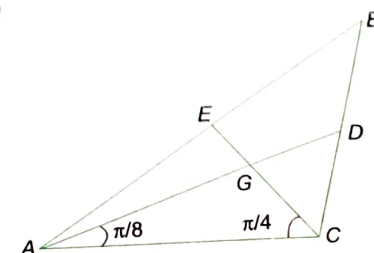
$$\therefore \cos B = \frac{1 - \tan^2 \frac{B}{2}}{1 + \tan^2 \frac{B}{2}} = \frac{1}{26}$$

$$\therefore \sin B = \frac{\sqrt{675}}{26} = \frac{15\sqrt{3}}{26}$$

$$\therefore \sin C = \sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{4\sqrt{3}}{13}$$

$$\text{Now } \frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow a = 13$$

52. (2)



Let G be the point of intersection of the medians of triangle ABC . Then the area of $\triangle ABC$ is three times that of $\triangle AGC$.

Now in $\triangle AGC$, $AG = \frac{2}{3} AD = \frac{10}{3}$.

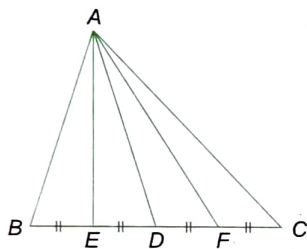
Applying the sine rule to $\triangle AGC$, we get

$$\frac{GC}{\sin(\pi/8)} = \frac{AG}{\sin(\pi/4)} \Rightarrow GC = \frac{10}{3} \frac{\sin(\pi/8)}{\sin(\pi/4)}$$

$$\begin{aligned} \text{Area of } \triangle AGC &= \frac{1}{2} AG \cdot GC \sin \angle AGC \\ &= \frac{1}{2} \cdot \frac{10}{3} \cdot \frac{10}{3} \frac{\sin(\pi/8)}{\sin(\pi/4)} \sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right) \\ &= \frac{50}{9} \frac{\sin(\pi/8) \cos(\pi/8)}{\sin(\pi/4)} \\ &= \frac{50}{18} = \frac{25}{9} \end{aligned}$$

$$\therefore \text{Area of } \triangle ABC = 3 \times \frac{25}{9} = \frac{25}{3}$$

53. (1)



$$\text{In } \triangle ABC, AD^2 = m_1^2 = \frac{c^2 + b^2}{2} - \frac{a^2}{4}$$

$$\text{In } \triangle ABD, AE^2 = m_2^2 = \frac{AD^2 + c^2}{2} - \left(\frac{a}{2}\right)^2$$

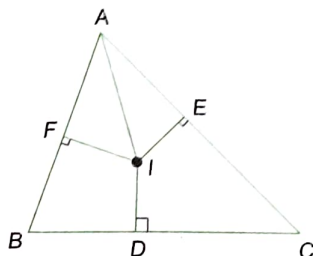
[Apollonius theorem]

$$\text{In } \triangle ADC, AF^2 = m_3^2 = \frac{AD^2 + b^2}{2} - \left(\frac{a}{2}\right)^2$$

$$\begin{aligned} \therefore m_2^2 + m_3^2 &= AD^2 + \frac{b^2 + c^2}{2} - \frac{a^2}{8} \\ &= m_1^2 + m_1^2 + \frac{a^2}{4} - \frac{a^2}{8} = 2m_1^2 + \frac{a^2}{8} \end{aligned}$$

$$\text{or } m_2^2 + m_3^2 - 2m_1^2 = \frac{a^2}{8}$$

54. (3)



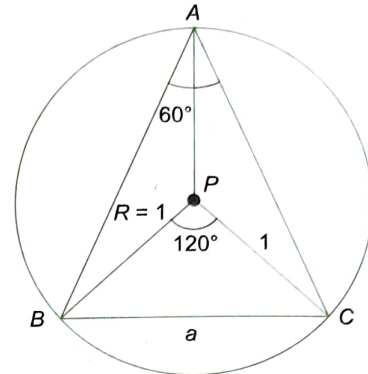
In triangles AIF and AIE ,

$$\frac{IF}{\sin(A/2)} = AI = \frac{IE}{\sin(A/2)}$$

$$\Rightarrow AI^2 = \frac{IE \cdot IF}{\sin^2(A/2)}$$

$$\Rightarrow \frac{ID \cdot IE \cdot IF}{IA \cdot IB \cdot IC} = \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R} = \frac{1}{10}$$

55. (1) Let R' be the radius of the circumcircle of $\triangle ABC$ using the sine rule in triangle BPC . Then



$$\frac{a}{\sin 120^\circ} = 2R_1 \quad \dots(i)$$

$$\text{Also, } \frac{a}{\sin 60^\circ} = 2R \quad (\text{in } \triangle ABC)$$

$$\therefore a = 2R \sin 60^\circ$$

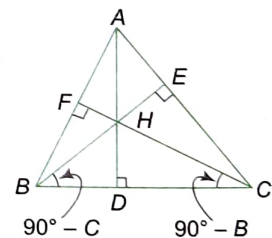
$$\text{or } a = \sqrt{3}$$

[$R = 1$, given]

$$\text{From Eq. (i), we get } R_1 = \frac{2\sqrt{3}}{\sqrt{3}} \times \frac{1}{2}$$

56. (4) Circumdiameter of $\triangle HBC$

$$= \frac{a}{\sin \angle BHC} = \frac{a}{\sin A} = 2R = 8$$



57. (2) The distance of circumcenter from side AC is $R \cos B$, and the distance of incenter from side AC is r . Therefore,

$$R \cos B = r \text{ or } \frac{r}{R} = \cos B$$

$$\Rightarrow 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \cos B$$

$$\Rightarrow \cos A + \cos B + \cos C - 1 = \cos B$$

$$\text{or } \cos A + \cos C = 1$$

58. (2) The distance of circumcenter from side AC is $R \cos B$, and the distance of orthocenter from side AC is $2R \cos A \cos C$. Therefore,

$$R \cos B = 2R \cos A \cos C$$

$$\therefore -\cos(A + C) = 2 \cos A \cos C$$

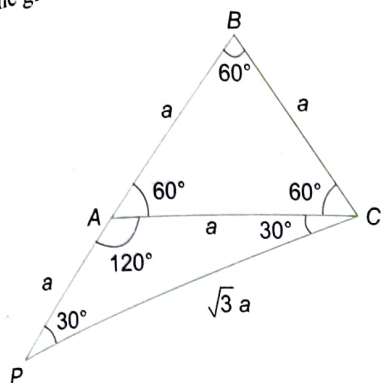
$$\therefore \sin A \sin C = 3 \cos A \cos C$$

$$\therefore \tan A \tan C = 3$$

$$CD = \frac{2ab}{a+b} \cos \frac{C}{2} = \frac{2ab}{a+b} \cos \frac{\pi}{3} = \frac{ab}{a+b} = 1$$

$$\therefore \frac{2ab}{a+b} = 2$$

(3) From the given information, we get the following figure.



$$\angle CAP = 120^\circ$$

So, in $\triangle PAC$, using cosine rule, we get

$$PC^2 = a^2 + a^2 - 2(a)^2 \left(\frac{-1}{2} \right) = 3a^2$$

In $\triangle PAC$,

$$\text{Perimeter} = s_1 = \frac{2a + \sqrt{3}a}{2}$$

Using formula for in radius, $\frac{A}{2}$, we get

$$r_1 = \frac{\sqrt{3}a}{2} \times \tan 15^\circ = \frac{\sqrt{3}a}{2} \times (2 - \sqrt{3})$$

In $\triangle PBC$, let r_2 be exradius.

$$\text{Perimeter, } s_2 = \frac{2a + a + \sqrt{3}a}{2}$$

$$r_2 = \frac{\sqrt{3}a}{2} (1 + \sqrt{3}) \tan 15^\circ \quad \left(\text{using } r_1 = s \tan \frac{A}{2} \right)$$

$$= \frac{\sqrt{3}a}{2} (1 + \sqrt{3}) (2 - \sqrt{3})$$

$$\therefore r_1 + r_2 = \frac{\sqrt{3}a}{2} (2 - \sqrt{3}) (1 + \sqrt{3} + 1) = \frac{\sqrt{3}a}{2}$$

$$61. (1) BD = (s - b), CD = (s - c)$$

$$\Rightarrow (s - b)(s - c) = 2$$

$$\text{or } s(s - a)(s - b)(s - c) = 2s(s - a)$$

$$\text{or } \Delta^2 = 2s(s - a)$$

$$\text{or } \frac{\Delta^2}{s^2} = \frac{2(s - a)}{s} \quad (\text{using } \Delta = rs)$$

$$\text{or } r^2 = \frac{2(s - a)}{s}$$

$$\text{or } \frac{a}{s} = \text{constant}$$

Now, $\Delta = \frac{1}{2} aH_a$, where H_a is the distance of A from BC . Thus,

$$\frac{\Delta}{s} = \frac{1}{2} \frac{aH_a}{s} = 1 \text{ or } H_a = \frac{2s}{a} = \text{constant}$$

Therefore, locus of A will be a straight line parallel to side BC .

$$62. (3) b \cot B + c \cot C = 2(r + R)$$

$$\Rightarrow 2R \sin B \cdot \frac{\cos B}{\sin B} + 2R \sin C \cdot \frac{\cos C}{\sin C} = 2(r + R)$$

$$\Rightarrow \cos B + \cos C = 1 + \frac{r}{R}$$

$$\Rightarrow \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\Rightarrow \cos B + \cos C = \cos A + \cos B + \cos C$$

$$\therefore \cos A = 0$$

$$\Rightarrow A = \frac{\pi}{2}$$

$$\Rightarrow a^2 = b^2 + c^2$$

$$\Rightarrow b^2 + c^2 = 100$$

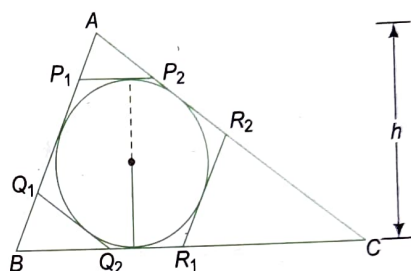
Using A.M. $\geq$ G.M., we get

$$\frac{b^2 + c^2}{2} \geq \sqrt{b^2 c^2}$$

$$\Rightarrow bc \leq 50$$

$$\text{Hence, area of } \triangle ABC = \frac{1}{2} bc \leq 25$$

63. (2)



$$\triangle AP_1P_2 \sim \triangle ABC$$

$$\Rightarrow \frac{t_1}{a} = \frac{h - 2r}{h} = 1 - \frac{2r}{h}$$

$$\Rightarrow \frac{t_1}{a} = 1 - \frac{2\Delta}{sh}$$

(where $\Delta \equiv \text{ar}(\triangle ABC)$)

$$\Rightarrow \frac{t_1}{a} = 1 - \frac{2 \cdot \frac{1}{2} ah}{sh}$$

$$\Rightarrow \frac{t_1}{a} = 1 - \frac{a}{s}$$

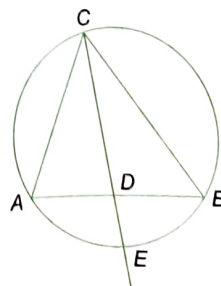
$$\text{Similarly, } \frac{t_2}{b} = 1 - \frac{b}{s} \text{ and } \frac{t_3}{c} = 1 - \frac{c}{s}$$

$$\therefore \frac{t_1}{a} + \frac{t_2}{b} + \frac{t_3}{c} = 3 - \frac{(a + b + c)}{s} = 3 - 2 = 1$$

$$64. (3) CD \cdot DE = AD \cdot DB$$

$$\text{But } \frac{AD}{BD} = \frac{b}{a}$$

$$\Rightarrow AD = \frac{bc}{a + b} \text{ and } DB = \frac{ac}{a + b}$$



$$\therefore CD \cdot DE = \frac{ab}{(a + b)^2} c^2$$

But c^2 is given, so we have to find the maximum value of

$$\frac{ab}{(a+b)^2}$$

$$\text{Now } \frac{ab}{(a+b)^2} = \frac{1}{\frac{a}{b} + \frac{b}{a} + 2}$$

$$\text{But } \frac{a}{b} + \frac{b}{a} \geq 2$$

$$\Rightarrow \frac{a}{b} + \frac{b}{a} + 2 \geq 4$$

$$\therefore \frac{ab}{(a+b)^2} \leq \frac{1}{4}$$

Hence, maximum value of $CD \cdot DE$ is $\frac{1}{4} c^2$.

$$65. (1) r_1 = 2r_2 = 3r_3$$

$$\Rightarrow \frac{\Delta}{s-a} = 2 \frac{\Delta}{s-b} = 3 \frac{\Delta}{s-c}$$

$$\text{or } \frac{1}{s-a} = \frac{2}{s-b} = \frac{3}{s-c} = k \text{ (say)}$$

$$\text{or } s-a = \frac{1}{k}, s-b = \frac{2}{k}, \text{ and } s-c = \frac{3}{k}$$

$$\text{Adding, we get } 3s - (a+b+c) = \frac{6}{k}$$

$$\text{or } s = \frac{6}{k} \quad \therefore a = \frac{5}{k}$$

$$\text{and } b = \frac{4}{k} \quad \text{or } \frac{a}{b} = \frac{5}{4}$$

$$66. (1) \text{ We have } \left(1 - \frac{s-b}{s-a}\right) \left(1 - \frac{s-c}{s-a}\right) = 2$$

$$\text{or } 2(b-a)(c-a) = 4(s-a)^2$$

$$\text{or } 2(bc - ac - ab + a^2) = (2s-2a)^2$$

$$\text{or } 2(bc - ac - ab + a^2) = (b+c-a)^2$$

$$\text{or } a^2 = b^2 + c^2$$

Hence, triangle is right angled.

$$67. (3) \frac{r}{r_1} = \frac{r_2}{r_3}$$

$$\text{or } r r_3 = r_1 r_2$$

$$\Rightarrow \frac{\Delta}{s} \frac{\Delta}{s-c} = \frac{\Delta}{s-a} \frac{\Delta}{s-b}$$

$$\text{or } \frac{(s-a)(s-b)}{s(s-c)} = 1$$

$$\Rightarrow \tan^2 \frac{C}{2} = 1 \text{ or } \tan \frac{C}{2} = 1$$

$$\text{or } \frac{C}{2} = 45^\circ \text{ or } C = 90^\circ$$

$$68. (3) \Delta_1 = \frac{1}{2} ra, \Delta_2 = \frac{1}{2} rb, \Delta_3 = \frac{1}{2} rc$$

Given that $\Delta_1, \Delta_2, \Delta_3$ are in A.P.

Therefore, a, b, c are in A.P.

Let Δ be the area of triangle ABC . Also, let x, y and z be the measures of the altitudes of the triangle.

$$\text{Then } x = \frac{2\Delta}{a}, y = \frac{2\Delta}{b}, z = \frac{2\Delta}{c}$$

Hence, x, y, z are in H.P.

$$69. (4) r - r_2 = r_3 - r_1$$

$$\Rightarrow \frac{\Delta}{s} - \frac{\Delta}{s-b} = \frac{\Delta}{s-c} - \frac{\Delta}{s-a}$$

$$\text{or } \frac{-b}{s(s-b)} = \frac{c-a}{(s-a)(s-c)}$$

$$\text{or } \frac{(s-a)(s-c)}{s(s-b)} = \frac{a-c}{b}$$

$$\Rightarrow \tan^2 \frac{B}{2} = \frac{a-c}{b}$$

$$\text{But } \frac{B}{2} \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right). \text{ Therefore,}$$

$$\tan^2 \frac{B}{2} \in \left(\frac{1}{3}, 1\right)$$

$$\Rightarrow \frac{1}{3} < \frac{a-c}{b} < 1$$

$$\text{or } b < 3a - 3c < 3b$$

$$\text{or } b + 3c < 3a < 3b + 3c$$

$$70. (1) \text{ We know that } II_1 = 4R \sin A/2$$

$$\text{Now, } \sum II_1 = 9$$

$$\Rightarrow 4R \sum \sin \frac{A}{2} = 9$$

$$\text{or } 4R \left(\frac{6}{5}\right) = 9 \text{ or } R = \frac{15}{8}$$

$$71. (1) \text{ Given } r_1, r_2, r_3 \text{ are in H.P. Thus,}$$

a, b, c are in A.P.

$$\text{and } a + b + c = 24 = 2s$$

$$\text{or } 2b + b = 24$$

$$\text{or } b = 8$$

$$\Rightarrow c = 16 - a$$

$$\text{Also given } \sqrt{s(s-a)(s-b)(s-c)} = 24$$

$$\text{or } \sqrt{12(12-a)(12-8)(a-4)} = 24$$

$$\text{or } (12-a)(a-4) = 12$$

$$\text{or } a^2 - 16a + 60 = 0 \Rightarrow a = 6, 10$$

$$72. (4) \text{ We have } r_1 - r = 4R \sin \frac{A}{2} \left[\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} \right]$$

$$\Rightarrow 7 - 1 = 4R \sin \frac{A}{2} \cos \left(\frac{B+C}{2} \right)$$

$$\Rightarrow 6 = 4R \sin^2 \frac{A}{2}$$

$$\Rightarrow 6 = 12 \sin^2 \frac{A}{2}$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{1}{2}$$

$$\Rightarrow \sin \frac{A}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{A}{2} = \frac{\pi}{4}$$

$$\Rightarrow A = \frac{\pi}{2}$$

So, PCP' is equilateral and thus, $PP' = 5$.

So, $\angle PAP' = 90^\circ$ [$\because AP = 3, PP' = 5$]

$$\therefore \text{Area of } \triangle APP' = \frac{1}{2} (3) (4) = 6$$

Multiple Correct Answers Type

1. (1), (3), (4)

$$(a^2 - 2ac + c^2) + (a^2 - 4ab + 4b^2) = 0$$

$$\text{or } (a - c)^2 + (a - 2b)^2 = 0$$

$$\Rightarrow a = c \text{ and } a = 2b$$

Therefore, the triangle is isosceles.

$$\text{Also, } \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7b^2}{8b^2} = \frac{7}{8}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{4}$$

2. (1), (3)

$$a + b > c \text{ or } 2b - c + b > c \text{ or } 3b > 2c$$

$$\text{or } \frac{b}{c} > \frac{2}{3}$$

$$\text{Also, } b + c > a \text{ or } b + c > 2b - c \text{ or } 2c > b$$

$$\text{or } \frac{b}{c} < 2$$

3. (2), (4)

Since $\sin A$ and $\sin B$ are the roots of $c^2x^2 - c(a+b)x + ab = 0$, we have

$$\sin A + \sin B = \frac{a+b}{c} \text{ and } \sin A \sin B = \frac{ab}{c^2}$$

$$\Rightarrow \frac{a}{2R} + \frac{b}{2R} = \frac{a+b}{c} \text{ and } \frac{a}{2R} \times \frac{b}{2R} = \frac{ab}{c^2}$$

$$\therefore c = 2R$$

$$\Rightarrow 2R \sin C = 2R$$

$$\Rightarrow \angle C = 90^\circ$$

$$\Rightarrow A + B = 90^\circ$$

$$\Rightarrow B = 90^\circ - A$$

$$\therefore \sin A + \sin B = \frac{a+b}{c}$$

$$\therefore x^2 + \frac{1}{x^2} \geq 2$$

4. (3), (4)

$$(1) \tan A + \tan B + \tan C = \tan A \tan B \tan C = 0$$

Therefore, either A or B or $C = 0$. This is not possible in a triangle.

$$(2) \text{ By sine rule, } \frac{a}{2} = \frac{b}{3} = \frac{c}{7} = \lambda \text{ (say)}$$

$$a + b = 5\lambda, c = 7\lambda, \text{ i.e., } a + b < c$$

This is not possible in a triangle, as the sum of two sides is greater than the third.

$$(3) \text{ Given that } (a+b)^2 = c^2 + ab$$

$$\text{or } a^2 + b^2 - c^2 + ab = 0$$

$$\text{or } 2ab \cos C + ab = 0$$

$$\text{or } \cos C = -\frac{1}{2} \text{ or } \angle C = 120^\circ$$

$$\text{Also } \sqrt{2} (\sin A + \cos A) = \sqrt{3}$$

$$\text{or } \sin(A + 45^\circ) = \frac{\sqrt{3}}{2}$$

(given)

$$\text{or } A + 45^\circ = 60^\circ$$

$$\text{or } A = 15^\circ \text{ and hence } B = 45^\circ$$

$$\therefore \triangle \text{ is possible}$$

$$(4) \text{ Given that } \cos A \cos B = \frac{\sqrt{3}}{4} = \sin A \sin B$$

$$\Rightarrow \cos(A - B) = \frac{\sqrt{3}}{2}$$

$$\text{or } A - B = \frac{\pi}{6}$$

$$\text{and } \cos(A + B) = 0$$

$$\text{or } A + B = \frac{\pi}{2}$$

$$\Rightarrow A = 60^\circ, B = 30^\circ$$

$$\text{Hence, } \sin A + \sin B = \frac{\sqrt{3} + 1}{2}. \text{ This is possible in a triangle.}$$

5. (1), (2)

$$2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$$

$$\text{Also, } (a^2 - b^2 + c^2)^2 = a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 - c^2a^2)$$

$$\therefore (a^2 - b^2 + c^2)^2 = 2c^2a^2$$

$$\therefore \frac{a^2 - b^2 + c^2}{2ca} = \pm \frac{1}{\sqrt{2}} = \cos B$$

$$\text{or } B = 45^\circ \text{ or } 135^\circ$$

6. (1), (3)

$$\text{From the cosine formula, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{or } b^2 - (2c \cos A) b + (c^2 - a^2) = 0,$$

which is a quadratic equation in b . Therefore,

$$c \sin A < a < c$$

Therefore, two triangles will be obtained. But this is possible when two values of the third side are also obtained. Clearly, two values of sides b will be b_1 and b_2 . Let these be the roots of the above equation. Then,

$$\text{Sum of roots} = b_1 + b_2 = 2c \cos A \text{ and } b_1 b_2 = c^2 - a^2.$$

7. (1), (2)

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= (a - b)^2 + 2ab(1 - \cos C)$$

$$= (a - b)^2 + \frac{4\Delta}{\sin C} (1 - \cos C)$$

For c to be minimum, $a = b$.

$$\text{Also, } \Delta = \frac{1}{2} ab \sin C$$

$$\Rightarrow a^2 = \frac{2\Delta}{\sin C} = b^2$$

8. (2), (3)

$$2\Delta = ab \sin C \Rightarrow a^2b^2 - 4\Delta^2 = a^2b^2 \cos^2 C$$

$$\Rightarrow \sqrt{a^2b^2 - 4\Delta^2} + \sqrt{b^2c^2 - 4\Delta^2} + \sqrt{c^2a^2 - 4\Delta^2}$$

$$= ab \cos C + bc \cos A + ca \cos B$$

$$= \frac{a^2 + b^2 + c^2}{2}$$

(using cosine rule)

9. (1), (4)

Sides are in A.P. and $a < \min\{b, c\}$

Therefore, order of A.P. can be b, c, a or c, b, a .

Case I:

If $2c = a + b$, then

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2c - b)^2}{2bc} = \frac{4b - 3c}{2b}$$

Case II:

If $2b = a + c$, then

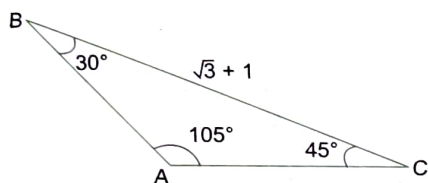
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b - c)^2}{2bc} = \frac{4c - 3b}{2c}$$

10. (1), (3)

$$\text{We have } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\text{or } \frac{\sin 105^\circ}{\sqrt{3} + 1} = \frac{\sin 30^\circ}{b} = \frac{\sin 45^\circ}{c}$$

$$\text{or } b = \frac{(\sqrt{3} + 1)}{2 \sin 105^\circ}, c = \frac{\sqrt{3} + 1}{\sqrt{2} \sin 105^\circ}$$



$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} bc \sin A = \frac{1}{2} bc \sin 105^\circ \\ &= \frac{1}{2} \cdot \frac{(\sqrt{3} + 1)^2}{2 \sqrt{2} \sin^2 105^\circ} \sin 105^\circ = \frac{\sqrt{3} + 1}{2} \end{aligned}$$

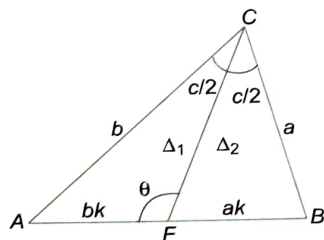
$$\text{Also, } \frac{a}{b} = \frac{\sin 105^\circ}{\sin 30^\circ} = \frac{\sqrt{3} + 1}{\sqrt{2}}$$

11. (1), (3), (4)

Given that $s - a$, $s - b$, and $s - c$ are in A.P. $\Rightarrow a, b, c$ are in A.P. $\Rightarrow \frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c}$ are in H.P. $\Rightarrow r_1, r_2, r_3$ are in H.P.

$$\begin{aligned} \text{Also, } \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{b^2 + c^2 - (2b - c)^2}{2bc} = \frac{4c - 3b}{2c} \end{aligned}$$

12. (1), (3)



$$\Delta = \Delta_1 + \Delta_2$$

$$\Rightarrow \frac{1}{2} ab \sin C = \frac{1}{2} b(CF) \sin \frac{C}{2} + \frac{1}{2} a(CF) \sin \frac{C}{2}$$

$$\text{or } CF = \frac{ab \sin C}{(a+b) \sin \frac{C}{2}} = \frac{2ab \cos \frac{C}{2}}{a+b}$$

Again in $\triangle CFB$, by the sine rule, we have

$$\frac{CF}{\sin B} = \frac{a}{\sin(\pi - \theta)} = \frac{a}{\sin \theta} = \frac{a}{\sin\left(B + \frac{C}{2}\right)}$$

$$\left(\because \theta + B + \frac{C}{2} = \pi \right)$$

$$\text{or } CF = \frac{a \sin B}{\sin\left(B + \frac{C}{2}\right)} = \frac{b \sin A}{\sin\left(B + \frac{C}{2}\right)} \quad [\text{by sine rule}]$$

13. (1), (3)

$$\begin{aligned} |BD - CD| &= |(s - b) - (s - c)| \\ &= |b - c| \\ &= 2R |\sin B - \sin C| \\ &= 4R \sin \frac{B - C}{2} \cdot \cos \frac{B + C}{2} \\ &= \left| 4R \sin \frac{B - C}{2} \cdot \sin \frac{A}{2} \right| \end{aligned}$$

14. (1), (2), (3), (4)

$$\tan \frac{B}{2} = \frac{2}{3}, \tan \frac{C}{2} = \frac{1}{2} \Rightarrow \tan \frac{A}{2} = \frac{4}{7};$$

$$\Rightarrow \tan \frac{B}{2} \tan \frac{C}{2} = \frac{s - a}{s} = \frac{1}{3}$$

$$\Rightarrow 2s = 3a = 42$$

$$\therefore \Delta = rs = 84 \text{ cm}^2$$

Thus, $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$ all are less than 1.

Hence, all angles of the triangle are acute.

15. (1), (2)

$$AH = 2R \cos A, BH = 2R \cos B, CH = 2R \cos C$$

$$\therefore P = 2R (\cos A + \cos B + \cos C)$$

$$= 2R \left(1 + \frac{r}{R} \right)$$

$$= 2(R + r)$$

We know that in any triangle, $r \leq \frac{R}{2}$

$$\therefore P \leq 2R + R$$

$$\Rightarrow P \leq 3R$$

16. (1), (2), (4)

$$r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s - a}$$

$$r_1 r = \frac{\Delta^2}{s(s - a)} = \frac{s(s - a)(s - b)(s - c)}{s(s - a)}$$

$$= (s - b)(s - c) = (s - b)^2 \quad (\because b = c)$$

$$= \frac{(2s - 2b)^2}{4}$$

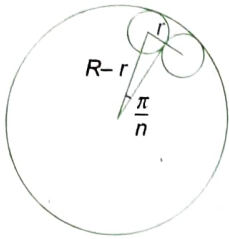
$$= \frac{(a + b + c - 2b)^2}{4} \quad (\because b = c)$$

$$= \frac{a^2}{4} = \frac{4R^2 \sin^2 A}{4} = R^2 \sin^2 A$$

Also if $\angle B = \theta \Rightarrow \angle A = \pi - 2\theta$

$$r_1 r = R^2 \sin^2(\pi - 2\theta) = R^2 \sin^2 2\theta = R^2 \sin^2 2B$$

17. (1), (4)



If R is radius of bigger circle, then $\sin\left(\frac{\pi}{n}\right) = \frac{r}{R-r}$

$$\text{or } R = r \left(1 + \operatorname{cosec}\left(\frac{\pi}{n}\right) \right) = \frac{r \left[\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} \right]^2}{\sin \frac{\pi}{n}}$$

18. (1), (2), (3)

Area of polygon = $n \times$ Area of $\triangle OBC$

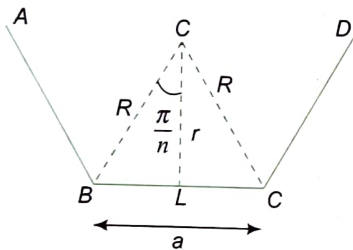
$$= \frac{na^2}{4} \cot \frac{\pi}{n} \quad (i)$$

$$\text{Now, } a = 2r \tan \frac{\pi}{n}$$

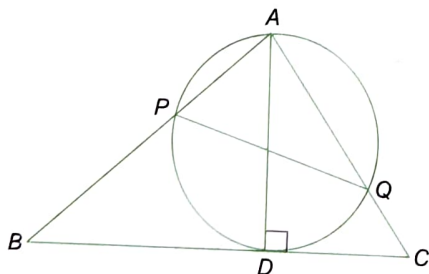
$$\Rightarrow \text{Area of polygon} = nr^2 \tan\left(\frac{\pi}{n}\right) \quad [\text{from Eq. (i)}]$$

$$\text{Also, } a = 2R \sin \frac{\pi}{n}$$

$$\Rightarrow = \frac{nR^2}{2} \sin\left(\frac{2\pi}{n}\right) \quad [\text{from Eq. (i)}]$$



19. (3), (4)



$$AD = c \sin B$$

Let R' be circumradius of $\triangle APQ$.

Using sine rule in triangle APQ , we get

$$\frac{PQ}{\sin A} = 2R' = AD = c \sin B$$

$$\Rightarrow PQ = c \sin B \sin A = c \frac{b}{2R} \sin A = \frac{\Delta}{R}$$

$$\text{Also, } PQ = c \sin B \sin A \\ = 2R \sin C \sin B \sin A$$

20. (1), (2)

We have $2s = a + b + c$, $A^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\geq$ G.M.

$$\Rightarrow \frac{s + (s-a) + (s-b) + (s-c)}{4}$$

$$\geq [s(s-a)(s-b)(s-c)]^{1/4}$$

$$\Rightarrow \frac{4s-2s}{4} \geq [A^2]^{1/4}$$

$$\Rightarrow \left(\frac{s}{2}\right) \geq A^{1/2} \quad \text{or} \quad A \leq \left(\frac{s^2}{4}\right)$$

$$\text{Also, } \frac{(s-a) + (s-b) + (s-c)}{3}$$

$$\geq [(s-a)(s-b)(s-c)]^{1/3}$$

$$\text{or } \frac{s}{3} \geq \left[\frac{A^2}{s}\right]^{1/3} \quad \text{or } \frac{A^2}{s} \leq \frac{s^3}{27} \quad \text{or } A \leq \frac{s^2}{3\sqrt{3}}$$

21. (1), (2), (3), (4)

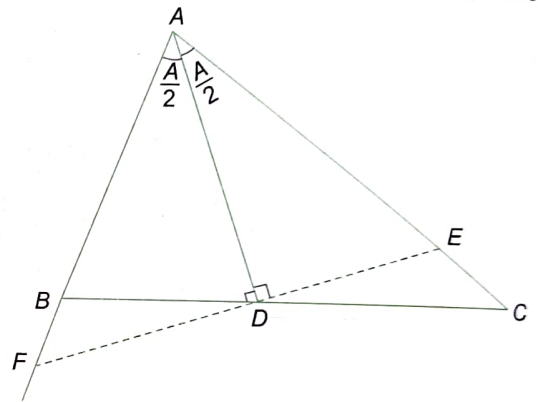
In $\triangle AFE$, AD is angle bisector of $\angle A$. and $AD \perp EF$.

$\therefore D$ is midpoint of EF and $\triangle AEF$ is isosceles triangle.

Therefore, $\triangle AFE$ is an isosceles triangle. Now,

To find AD ,

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle ABD) + \text{Area}(\triangle ADC)$$



$$\Rightarrow \frac{1}{2} bc \sin A = \frac{1}{2} c AD \sin \frac{A}{2} + \frac{1}{2} b AD \sin \frac{A}{2}$$

$$\Rightarrow AD = \frac{2bc \cos \frac{A}{2}}{b+c}$$

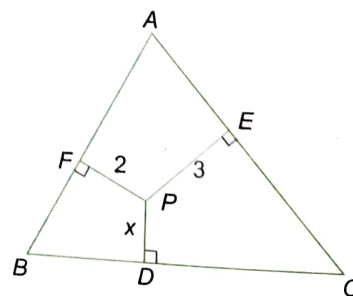
$$\text{Also, } AD = AE \cos \frac{A}{2}$$

(From $\triangle ADE$)

$$\Rightarrow AE = \frac{2bc}{b+c} = \text{H.M. of } b \text{ and } c$$

$$\text{Again } EF = 2 DE = 2 AD \tan \frac{A}{2} = \frac{4bc \sin \frac{A}{2}}{b+c}$$

22. (1), (2), (3)



$$a = 7, b = 6, c = 5.$$

$$\therefore s = 9$$

$$\therefore \text{Area } \Delta = \sqrt{s(s-a)(s-b)(s-c)} = 6\sqrt{6}$$

Now, $PF = 2$, $PE = x$.

Let $PD = x$

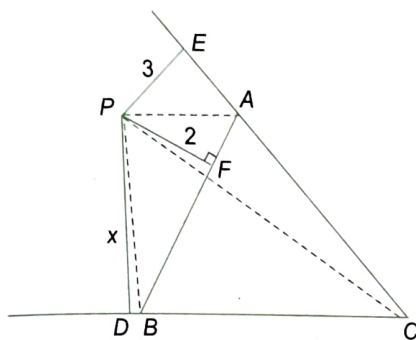
Then

$$\Delta = \frac{1}{2} \times 2 \times 5 + \frac{1}{2} \times x \times 7 + \frac{1}{2} \times 3 \times 6$$

$$\therefore 6\sqrt{6} = 5 + \frac{7x}{2} + 9$$

$$\therefore x = \frac{12\sqrt{6} - 28}{7}$$

When P lies outside the triangle



$$\text{Area of } \Delta ABC = \frac{1}{2} x \times 7 + \frac{1}{2} \times 3 \times 6 - \frac{1}{2} \times 2 \times 5$$

$$\therefore 6\sqrt{6} = \frac{7}{2}x + 9 - 5$$

$$\therefore x = \frac{12\sqrt{6} - 8}{7}$$

Similarly, in one more case, we get $\frac{12\sqrt{6} + 8}{7}$.

23. (2), (4)

$$\text{Given, } \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta} = \frac{2s(s-a)}{\Delta}$$

$$\Rightarrow s-b+s-c=2(s-a)$$

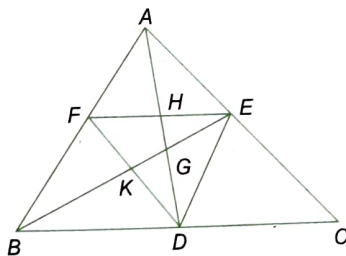
$$\Rightarrow b+c=2a$$

So, locus of vertex A is an ellipse.

24. (1), (2)

Let D, E and F be the mid-points of the sides of ΔABC .

E and F are mid-points of AC and AB , respectively.



So, $EF \parallel BC$ and $EF = \frac{1}{2} BC$

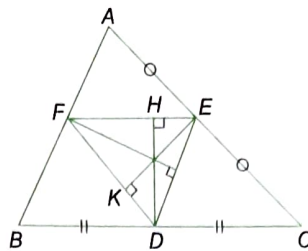
Thus, AD will also divide EF into equal parts.

Hence, DH is also median of ΔDEF .

Similarly BE , median of ΔABC , is also median of ΔDEF .

These two lines meet at G . So the centroids of both triangles are same.

The orthocenter, O of ΔDEF is the point of intersection of the perpendiculars DH and EK drawn from D and E , respectively.



Since, $EF \parallel BC$, DH is perpendicular to BC also.

Similarly EK is perpendicular to AC .

So, orthocenter of ΔDEF and circumcentre of ΔABC is the same points.

Linked Comprehension Type

1. (1), 2. (2), 3. (3)

We have,

$$r_1 = \frac{\Delta}{s-a} = 2, r_2 = \frac{\Delta}{s-b} = 3, r_3 = \frac{\Delta}{s-c} = 6$$

Given $\Delta = 6$,

$$\therefore s-a=3 \quad \dots(i)$$

$$s-b=2 \quad \dots(ii)$$

$$s-c=1 \quad \dots(iii)$$

Adding Eqs. (i) and (ii),

$$2s-a-b=5 \quad \text{or} \quad a+b+c-a-b=5 \quad \text{or} \quad c=5$$

Adding Eqs. (i) and (iii), $2s-a-c=4$, or $b=4$

And adding Eqs. (ii) and (iii), $2s-b-c=3$, or $a=3$

Hence the sides of the Δ are $a=3, b=4, c=5$.

Since the triangle is right angled, the greatest angle is 90° .

Also, the least angle is opposite to side a , which is

$\sin^{-1} \frac{3}{5}$. Therefore,

$$90^\circ - \sin^{-1} \frac{3}{5} = \cos^{-1} \frac{3}{5}$$

$$\text{Also, } R = \frac{abc}{4\Delta} = \frac{60}{24} = 2.5$$

$$r = \frac{\Delta}{s} = \frac{6}{6} = 1$$

4. (1) 5. (3) 6. (4)

$$\cos(A-B) = \frac{4}{5}$$

$$\Rightarrow \frac{1 - \tan^2 \frac{A-B}{2}}{1 + \tan^2 \frac{A-B}{2}} = \frac{4}{5}$$

$$\text{or } \tan^2 \frac{A-B}{2} = \frac{1}{9}$$

$$\text{or } \tan \frac{A-B}{2} = \frac{1}{3}$$

$$\text{Now, } \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\text{or } \frac{1}{3} = \frac{6-3}{6+3} \cot \frac{C}{2}$$

$$\text{or } \cot \frac{C}{2} = 1 \quad \text{or } C = \frac{\pi}{2}$$

$$\text{Area of triangle} = \frac{1}{2} ab \sin C = \frac{1}{2} \times 6 \times 3 \times 1 = 9$$

$$\frac{a}{\sin A} = \frac{\sqrt{a^2 + b^2}}{1}$$

$$\text{or } \frac{6}{\sin A} = \sqrt{45}$$

$$\text{or } \sin A = \frac{2}{\sqrt{5}}$$

7. (1) 8. (2), 9. (2)

$$AH = 2R \cos A$$

$$BH = 2R \cos B$$

$$CH = 2R \cos C$$

$$HD = 2R \cos B \cos C$$

$$HE = 2R \cos A \cos C$$

$$HF = 2R \cos A \cos B$$

$$\text{Now } AH \cdot BH \cdot CH = 3$$

(given)

$$\Rightarrow \cos A \cdot \cos B \cdot \cos C = \frac{3}{8R^3} \quad \dots(i)$$

$$\text{Now } AH^2 + BH^2 + CH^2 = 7$$

(given)

$$\Rightarrow 4R^2 \sum \cos^2 A = 7$$

$$\text{or } \sum \cos^2 A = \frac{7}{4R^2}$$

Now we know

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$$

$$\frac{7}{4R^2} = 1 - 2 \times \frac{3}{8R^3}$$

$$\text{or } 4R^3 - 7R - 3 = 0$$

$$\text{or } (R+1)(2R+1)(2R-3) = 0$$

$$\text{or } R = \frac{3}{2}$$

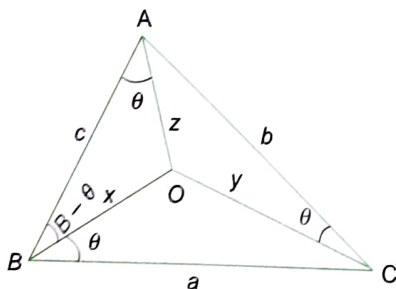
Now $HD \cdot HE \cdot HF$

$$= (2R \cos B \cos C)(2R \cos A \cos C)(2R \cos A \cos B)$$

$$= 8R^3 \cos^2 A \cos^2 B \cos^2 C$$

$$= 8R^3 \times \frac{9}{64R^6} = \frac{9}{8R^3}$$

10. (4), 11. (2), 12. (1)



Applying sine rule in $\triangle AOB$, we have

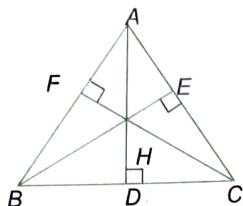
$$\frac{OA}{\sin \angle ABO} = \frac{AB}{\sin \angle AOB}$$

$$\text{or } OA = \frac{c \sin \angle ABO}{\sin \angle AOB} = \frac{c \sin (B - \theta)}{\sin B} \quad \dots(i)$$

$$[\because \angle ABO = B - \theta, \angle AOB = 180^\circ - \theta - \angle ABO = 180^\circ - B]$$

Again in $\triangle AOC$, we have

$$\frac{OA}{\sin \angle ACO} = \frac{AC}{\sin \angle AOC}$$



$$\Rightarrow OA = \frac{b \sin \angle ACO}{\sin \angle AOC} = \frac{b \sin \theta}{\sin A} \quad \dots(ii)$$

$$[\because \angle OAC = A - \theta, \angle AOC = 180^\circ - \theta - \angle OAC = 180^\circ - A]$$

From Eqs. (i) and (ii), we have

$$\frac{c \sin (B - \theta)}{\sin B} = \frac{b \sin \theta}{\sin A}$$

$$\text{or } c \sin A \sin (B - \theta) = b \sin \theta \sin B$$

$$= b \sin \theta \sin (A + C)$$

$$\text{or } 2R \sin C \sin A (\sin B \cos \theta - \cos B \sin \theta) = 2R \sin B \sin \theta (\sin A \cos C + \cos A \sin C)$$

Dividing both sides by $2R \sin \theta \sin A \sin B \sin C$, we get

$$\cot \theta - \cot B = \cot C + \cot A$$

$$\text{or } \cot \theta = \cot A + \cot B + \cot C \quad \dots(iii)$$

Squaring both sides, we have

$$\cot^2 \theta = \cot^2 A + \cot^2 B + \cot^2 C$$

$$+ 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)$$

$$\text{or } \operatorname{cosec}^2 \theta - 1 = (\operatorname{cosec}^2 A - 1) + (\operatorname{cosec}^2 B - 1) + (\operatorname{cosec}^2 C - 1) + 2$$

$$[\text{since in } \triangle ABC, \cot A \cot B + \cot B \cot C + \cot C \cot A = 1]$$

$$\text{or } \operatorname{cosec}^2 \theta = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

Area of triangle ABC ,

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3$$

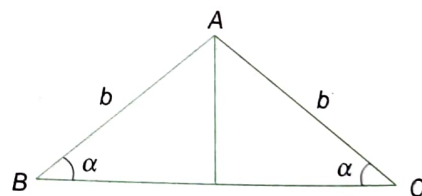
$$= \frac{1}{2} [aOB + bOC + cOA] \sin \theta$$

$$= \frac{1}{4} \tan \theta [2aOB \cos \theta + 2bOC \cos \theta + 2cOA \cos \theta]$$

$$= \frac{1}{4} \tan \theta [(a^2 + x^2 - y^2) + (b^2 + y^2 - z^2) + (c^2 + z^2 - x^2)]$$

$$= \frac{1}{4} \tan \theta [a^2 + b^2 + c^2]$$

13. (1), 14. (2), 15. (1)



$$\text{From sine rule, } \frac{b}{\sin \alpha} = 2R \quad \text{or } R = \frac{b}{2} \operatorname{cosec} \alpha$$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2} b^2 \sin (\pi - 2\alpha)}{\frac{1}{2} (b + b + 2b \cos \alpha)} = \frac{b \sin 2\alpha}{2(1 + \cos \alpha)}$$

Distance of incenter (I) from the side BC is r .

Distance of circumcenter (O) from the side BC is

$$|R \cos A| = |R \cos (\pi - 2\alpha)| = R \cos 2\alpha$$

Hence, distance between circumcenter and incenter is given by

$$OI = |r + R \cos 2\alpha| \quad (\because \text{circumcenter lies outside } \Delta)$$

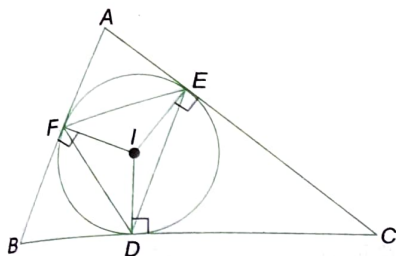
$$= \left| \frac{b \sin 2\alpha}{2(1 + \cos \alpha)} + \frac{b \operatorname{cosec} \alpha \cos 2\alpha}{2} \right|$$

$$= b \left| \frac{\sin 2\alpha}{4 \cos^2 \frac{\alpha}{2}} + \frac{\cos 2\alpha}{4 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \right|$$

$$= b \left| \frac{\cos 2\alpha \cos \frac{\alpha}{2} + \sin 2\alpha \sin \frac{\alpha}{2}}{4 \sin \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}} \right|$$

$$= \left| \frac{b \cos \left(3 \frac{\alpha}{2} \right)}{2 \sin \alpha \cos \frac{\alpha}{2}} \right|$$

(1), 17. (2), 18. (4)



Points I, D, C, E are concyclic. Therefore,
 $\angle EID = \pi - C$

Also, I is circumcenter of $\triangle DEF$. Hence,

$$\angle DFE = \frac{1}{2} \angle DIE = \frac{\pi - C}{2}$$

Similarly, $\angle FDE = \frac{\pi - A}{2}$

$$\angle FED = \frac{\pi - B}{2}$$

$$\therefore \text{Area of } \triangle DEF = 2r^2 \sin(\angle D) \sin(\angle E) \sin(\angle F)$$

$$= 2r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

Also by sine rule, in $\triangle DEF$,

$$\frac{EF}{\sin \angle D} = 2r$$

$$\text{or } EF = 2r \cos \frac{A}{2}$$

(1), 20. (4), 21. (2)

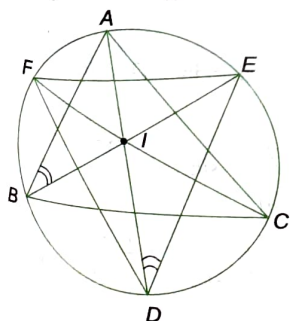
$$\text{Area of } \triangle ABC = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C$$

$$\angle ADE = \angle ABE = \frac{B}{2}$$

$$\text{Similarly, } \angle FDA = \angle FCA = \frac{C}{2}$$

$$\Rightarrow \angle FDE = \frac{B+C}{2}$$

$$\text{and } \angle DEF = \frac{A+C}{2} \text{ and } \angle DFE = \frac{A+B}{2}$$



In $\triangle DEF$, by sine rule, $\frac{EF}{\sin(\angle FDE)} = 2R$

$$\Rightarrow EF = 2R \cos \left(\frac{A}{2} \right)$$

Then, area of $\triangle DEF$

$$= 2R^2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{B+C}{2} \right) \sin \left(\frac{A+C}{2} \right)$$

$$= 2R^2 \cos \left(\frac{A}{2} \right) \cos \left(\frac{B}{2} \right) \cos \left(\frac{C}{2} \right)$$

$$\text{Now } \frac{\Delta_{ABC}}{\Delta_{DEF}} = \frac{2R^2 \sin A \sin B \sin C}{2R^2 \cos \left(\frac{A}{2} \right) \cos \left(\frac{B}{2} \right) \cos \left(\frac{C}{2} \right)}$$

$$= 8 \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right) \sin \left(\frac{C}{2} \right) \leq 1$$

$$\Rightarrow \Delta_{ABC} \leq \Delta_{DEF}$$

22. (1) Using A.M. $\geq$ G.M. for $s-a, s-b, s-c, s-d$, we have

$$\frac{s-a+s-b+s-c+s-d}{4} \geq \sqrt{A}$$

$$\text{or } \frac{4s-2s}{4} \geq 1$$

$$\text{or } 2s \geq 4$$

23. (1) If the length of diagonals are d_1 and d_2 , then the area of quadrilateral is given by

$$\text{Area} = \frac{1}{2} d_1 d_2 \sin \theta, \text{ where } \theta \text{ is angle between diagonals}$$

$$d_1 d_2 = \frac{2}{\sin \theta} \text{ or } d_1 d_2 \geq 2$$

$$\text{Also, } d_1 + d_2 \geq 2 \sqrt{d_1 d_2} \geq 2\sqrt{2}$$

24. (1) When the perimeter is minimum,

$$s-a=s-b=s-c=s-d \text{ or } a=b=c=d$$

Therefore, $ABCD$ is a square. ($\because$ cyclic quadrilateral)

$$25. (4) \frac{\Delta}{s-a} : \frac{\Delta}{s-b} : \frac{\Delta}{s-c} = 1 : 2 : 3$$

$$\text{Let } \frac{\Delta}{s-a} = \frac{\Delta}{s-b} = \frac{\Delta}{s-c} = \frac{\Delta}{6k}$$

$$\Rightarrow \frac{1}{s-a} = \frac{1}{6k}, \frac{1}{s-b} = \frac{1}{3k}, \frac{1}{s-c} = \frac{1}{2k}$$

$$\Rightarrow s-a=6k, s-b=3k, s-c=2k$$

$$\Rightarrow s=11k$$

$$\therefore a=5k, b=8k, c=9k$$

Hence, ratio of sides is $5 : 8 : 9$.

26. (1) Area of the triangle, $\Delta = k^2 \sqrt{11 \times 6 \times 3 \times 2} = 6\sqrt{11} k^2$

$$\text{So, } r = \frac{\Delta}{s} = \frac{6}{\sqrt{11}} k$$

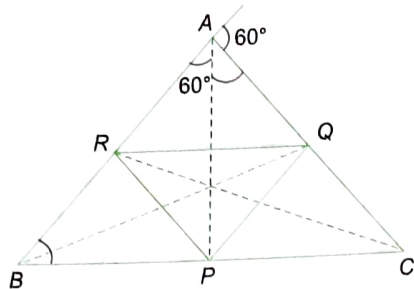
$$\text{And, } R = \frac{abc}{4\Delta} = \frac{5 \times 8 \times 9}{4 \times 6 \times \sqrt{11}} k = \frac{15}{\sqrt{11}} k$$

$$\therefore R : r = 5 : 2$$

27. (3) Since c is the greatest side, C is the greatest angle.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{25 + 64 - 81}{2 \times 5 \times 8} = \frac{1}{10}$$

28. (2) Since C is the excentre of $\triangle ABP$, we have.



$$\angle APQ = \angle QPC$$

Also, R is the excentre of $\triangle APC$

$$\text{so, } \angle APR = \angle RPS$$

$$\text{so, } \angle RPQ = 90^\circ$$

$$29. (4) \cos C = \left(\frac{a^2 + b^2 - c^2}{2ab} \right) = \frac{1}{2}$$

In $\triangle PQC$, we have

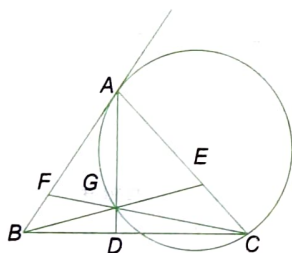
$$\cos C = \frac{CQ^2 + PC^2 - PQ^2}{2PC \cdot CQ}$$

$$\Rightarrow PQ^2 = PC^2 + CQ^2 - PC \cdot CQ$$

$$\left\{ \text{where, } PC = \frac{ab}{b+c} \text{ and } CQ = \frac{ab}{a+c} \right\}$$

$$\text{so } PQ = \sqrt{\frac{100}{9} + \frac{64}{9} - \frac{80}{9}} = \frac{\sqrt{84}}{3} = \frac{2}{3}\sqrt{21}$$

30. (3)



$$FG \times FC = FA^2$$

$$\Rightarrow \frac{1}{3}(FC^2) = FA^2$$

$$\Rightarrow \frac{2a^2 + 2b^2 - c^2}{3 \times 4} = \frac{c^2}{4} \quad (\text{Using Apollonius theorem})$$

$$\Rightarrow 2c^2 = a^2 + b^2.$$

31. (2) In $\triangle ADC$, we have,

$$\frac{a/2}{\sin(\pi/3)} = \frac{AD}{\sin C} \Rightarrow \sin C = \frac{\sqrt{3}}{2a} \sqrt{2b^2 + 2c^2 - a^2}$$

$$= \frac{\sqrt{3}}{2a} \sqrt{3b^2} = \frac{3}{2} \frac{b}{a} = \frac{1}{2}$$

$$32. (1) \text{ We have } AD = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}} = \frac{\sqrt{3}b}{2}$$

$$\text{If } b = 1 \text{ then } AD = \frac{\sqrt{3}}{2}$$

33. (2) 34. (3)

Let a, b and c ($a < b < c$) be the sides of given triangle.

$$\text{Also, } 2r = a + b - c$$

When $r = 4$ then, $(a, b) = (9, 40), (10, 24), (12, 16)$

$$\therefore \text{Greatest perimeter} = 9 + 40 + 41 = 90 \text{ units}$$

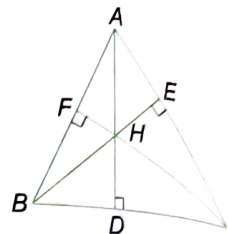
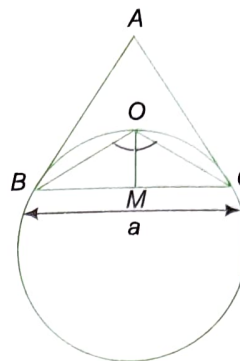
When $r = 5$ then $(a, b) = (11, 60), (12, 35), (15, 20)$
 $\therefore \text{Greatest area} = \frac{11 \times 60}{2} = 330 \text{ sq. unit}$

Matrix Match Type

1. $a \rightarrow q, b \rightarrow s, c \rightarrow s, d \rightarrow p.$

For solution, see the theory of ambiguous cases.

2. $a \rightarrow p, q; b \rightarrow r, s; c \rightarrow p, q; d \rightarrow p, q.$



O is circumcentre, then in $\triangle OMB$, $OM = R \cos A$ (distance of circumcentre from BC).

Similarly, distances of circumcentre from AC and AB are $R \cos B$ and $R \cos C$.

Applying sine rule in triangle OBC , we have

$$2R_1 = \frac{a}{\sin 2A} = \frac{a}{2 \sin A \cos A} = \frac{R}{\cos A}$$

$$\text{or } R_1 = \frac{R}{2 \cos A}$$

$$\text{Similarly, } R_2 = \frac{R}{\cos B} \text{ and } R_3 = \frac{R}{\cos C}$$

[where R_1, R_2, R_3 are circumradius of $\triangle OBC, \triangle OAB$, respectively]

H is orthocentre, then in $\triangle AFH$,

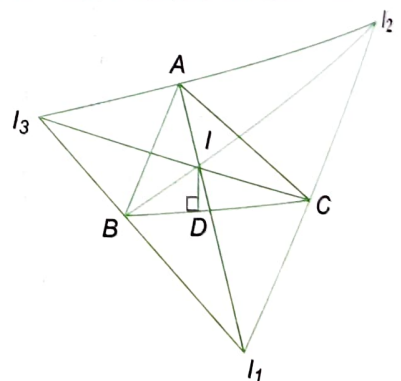
$$AH = \frac{AF}{\cos \angle FAH} = \frac{b \cos A}{\cos(90^\circ - B)} = 2R \cos A$$

Similarly, $BH = 2R \cos B, CH = 2R \cos C$.

Also, $HF = AF \tan \angle FAH$

$$= b \cos A \cot B = 2R \cos A \cos B$$

Similarly, $HE = 2R \cos A \cos C$ and $HD = 2R \cos B \cos C$



I is incentre, then in $\triangle IDB$, $BI = \frac{ID}{\sin \angle IDB} = \frac{r}{\sin(B/2)}$

$$\text{Similarly, } CI = \frac{r}{\sin(C/2)} \text{ and } AI = \frac{r}{\sin(A/2)}$$

Also in $\triangle BII_1$,

$$II_1 = \frac{BI}{\cos \angle BII_1} = \frac{r / [\sin(B/2)]}{\cos(\frac{\pi}{2} - \frac{C}{2})} = \frac{r}{\sin(B/2) \cos(C/2)}$$

$$\text{Similarly, } II_2 = \frac{r}{\sin(A/2)\sin(C/2)}$$

$$\text{and } II_3 = \frac{r}{\sin(A/2)\sin(B/2)}$$

3. $a \rightarrow p; b \rightarrow q; c \rightarrow p; d \rightarrow p, q.$

a. $\sin A, \sin B$ are roots of $c^2x^2 - c(a+b)x + ab = 0$

$$\therefore \sin A + \sin B = \frac{c(a+b)}{c^2}$$

$$\Rightarrow \frac{(a+b)}{2R} = \frac{c(a+b)}{c^2} \quad \text{or } 2R = c \quad \dots(i)$$

From sine formula, we have

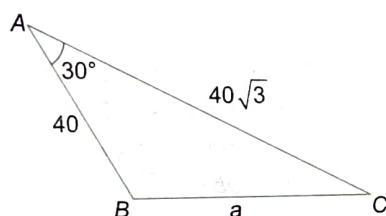
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = c \quad [\text{using Eq. (i)}]$$

$$\text{or } \sin C = 1 \quad \text{or } C = 90^\circ$$

b. Using cosine formula, we have

$$\cos 30^\circ = \frac{40^2 + (40\sqrt{3})^2 - a^2}{2 \times 40 \times 40\sqrt{3}}$$

$$\text{or } a = 40$$



Thus, $AB = BC = 40$, i.e., $\triangle ABC$ is isosceles.

Also, $\angle A = \angle C = 30^\circ$

$$\Rightarrow \angle B = 120^\circ$$

Therefore, $\triangle ABC$ is an obtuse-angled triangle.

$$c. 81^{\sin^2 x} + 81^{\cos^2 x} = 30$$

$$\text{or } 81^{\sin^2 x} + \frac{81}{81^{\sin^2 x}} = 30$$

$$\text{Put } 81^{\sin^2 x} = t$$

$$t + \frac{81}{t} = 30$$

$$\text{or } t = 27, t = 3$$

$$\text{When } t = 3, 81^{\sin^2 x} = 3$$

$$\text{or } 3^{4\sin^2 x} = 3$$

$$\text{or } x = 30^\circ$$

$$\text{When } t = 27, 3^{4\sin^2 x} = 3^3$$

$$\Rightarrow x = 60^\circ$$

Therefore, the triangle is right angled.

d. In $\triangle ABC$,

$$\cos A \cos B + \sin A \sin B \sin C = 1$$

$$\Rightarrow \sin C = \frac{1 - \cos A \cos B}{\sin A \sin B} \quad \dots(i)$$

$[\because \angle C \text{ is an angle of the triangle}]$

Since $0 < \sin C \leq 1$, we have

$$0 < \frac{1 - \cos A \cos B}{\sin A \sin B} \leq 1$$

$$\text{or } 1 - \cos A \cos B \leq \sin A \sin B$$

$$\text{or } 1 \leq \sin A \sin B + \cos A \cos B$$

$$\text{or } 1 \leq \cos(A - B) \text{ or } \cos(A - B) \geq 1$$

$$\text{or } \cos(A - B) = 1$$

$$\text{or } A - B = 0 \quad \text{or } A = B$$

Also from Eq. (i) and (ii),

$$\sin C = \frac{1 - \cos^2 A}{\sin^2 A}$$

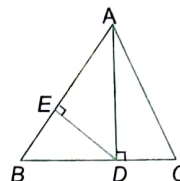
$$\sin C = 1 \quad \text{or } C = 90^\circ$$

From Eqs. (ii) and (iii), $\triangle ABC$ is a right-angled isosceles triangle.

4. $a \rightarrow q, b \rightarrow q, c \rightarrow p, d \rightarrow r.$

See the theory of different centers.

5. (4)



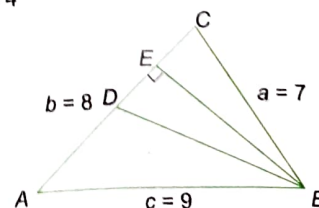
$$\begin{aligned} \text{a. Area of } \triangle ADB &= \frac{1}{2} AD \times BD \\ &= \frac{1}{2} c \sin B \times c \cos B \\ &= \frac{c^2}{4} \sin 2B \end{aligned}$$

$$\begin{aligned} \text{b. Area of } \triangle ADC &= \frac{1}{2} AD \times CD \\ &= \frac{1}{2} b \sin C \times b \cos C = \frac{b^2}{4} \sin 2C \end{aligned}$$

$$\begin{aligned} \text{c. Area of } \triangle ADE &= \frac{1}{2} AE \times DE \\ &= \frac{1}{2} AD \cos\left(\frac{\pi}{2} - B\right) AD \sin\left(\frac{\pi}{2} - B\right) \\ &= \frac{1}{4} AD^2 \sin 2B \\ &= \frac{1}{4} c^2 \sin^2 B \sin 2B \end{aligned}$$

$$\begin{aligned} \text{d. Area of } \triangle BDE &= \frac{1}{2} BE \times DE \\ &= \frac{1}{2} BD \cos B \times AD \sin\left(\frac{\pi}{2} - B\right) \\ &= \frac{1}{2} c \cos B \cos B \times c \sin B \cos B \\ &= \frac{1}{4} c^2 \cos^2 B \sin 2B \end{aligned}$$

6. (3) $AD = 4$



In $\triangle ABD$, using Cosine Rule

$$\cos A = \frac{4^2 + 9^2 - BD^2}{2 \times 4 \times 9}$$

In $\triangle ABC$, Using Cosine Rule

$$\cos A = \frac{8^2 + 9^2 - 7^2}{2 \times 8 \times 9}$$

$$\Rightarrow BD^2 = 49$$

$$\Rightarrow BD = 7$$

$\triangle BCD$ is isosceles.

$$\Rightarrow ED = CD/2 = 2$$

$$BE = \sqrt{7^2 - 2^2} = \sqrt{45}$$

$$AE = 6$$

Numerical Value Type

1. (1) Interior angle of regular polygon of side n is $\left(180^\circ - \frac{360^\circ}{n}\right)$.

Hence, $\alpha = 108^\circ$; $\beta = 120^\circ$; $\gamma = 144^\circ$; $\delta = 150^\circ$

$$\therefore \cos \alpha = \cos 108^\circ = -\sin 18^\circ = -\left(\frac{\sqrt{5}-1}{4}\right)$$

$$\sec \beta = \sec 120^\circ = -2$$

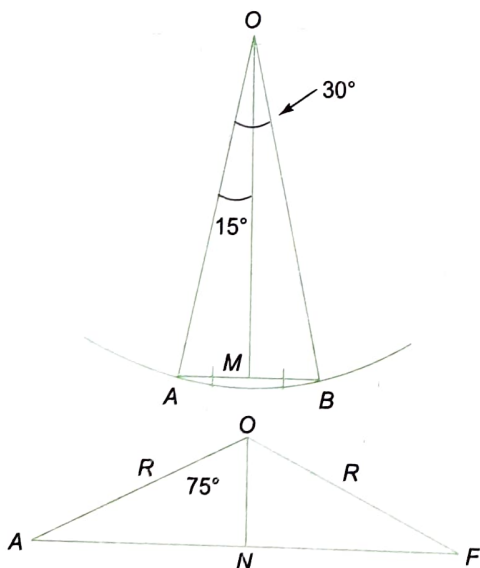
$$\cos \gamma = \cos 144^\circ = -\cos 36^\circ = -\left(\frac{\sqrt{5}+1}{4}\right)$$

$$\operatorname{cosec} \delta = \operatorname{cosec} 150^\circ = +2$$

$$\therefore \left| \left(\frac{\sqrt{5}-1}{4}\right)(2)\left(\frac{\sqrt{5}+1}{4}\right)(-2) \right| = 1$$

2. (4) $AM = R \sin \frac{\pi}{12}$ or $AB = 2R \sin \frac{\pi}{12}$

Similarly, $AN = R \sin \frac{5\pi}{12} \Rightarrow AF = 2R \sin \frac{5\pi}{12}$



$$\therefore \frac{AB}{AF} + \frac{AF}{AB} = \frac{\sin \frac{\pi}{12}}{\sin \frac{5\pi}{12}} + \frac{\sin \frac{5\pi}{12}}{\sin \frac{\pi}{12}}$$

$$= \tan \frac{\pi}{12} + \cot \frac{\pi}{12}$$

$$= (2 - \sqrt{3}) + (2 + \sqrt{3}) = 4$$

$$3. (8) AI = r \operatorname{cosec} \frac{A}{2} = \frac{\Delta}{s} \operatorname{cosec} \frac{A}{2}$$

$$\text{Also, } \Delta = \frac{1}{2} bc \sin A$$

$$\Rightarrow 30 = \frac{1}{2} \times 12 \times 5 \times \sin A$$

$$\Rightarrow \sin A = 1$$

$$\Rightarrow A = \frac{\pi}{2}$$

$$\therefore a^2 = b^2 + c^2$$

$$\Rightarrow a = 13$$

$$\therefore s = \frac{a+b+c}{2} = 15$$

Hence, $AI = 2\sqrt{2}$ units

$$4. (24) r = \frac{\Delta}{s}$$

$$s = 5 \text{ or } a+b+c = 10$$

$$\Delta = \frac{abc}{4R} \text{ or } abc = 60$$

$$\text{Now } \Delta^2 = s(s-a)(s-b)(s-c)$$

$$\text{or } 5 = (5-a)(5-b)(5-c)$$

$$= 125 - 25(a+b+c) + 5(ab+bc+ca) - abc$$

$$\therefore ab+bc+ca = 38$$

$$\text{or } a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 24$$

$$5. (84) \tan C = \frac{4}{3} \Rightarrow \cos C = \frac{3}{5}$$

$$a = n+1, b = n+2, c = n$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \frac{3}{5} = \frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)}$$

$$\text{or } n = 13$$

Hence, $a = 14$, $b = 15$, $c = 13$, and $s = 21$

$$\Rightarrow \Delta = \frac{1}{2} ab \sin C$$

$$= \left(\frac{1}{2}\right)(14)(15)\left(\frac{4}{5}\right)$$

$$= 84$$

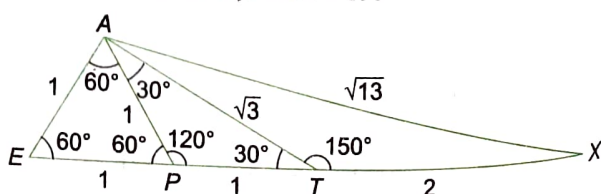
6. (13) We have $AE = EP = AP = 1$

$$\Rightarrow AP = PT = 1$$

Hence, $\triangle APT$ is isosceles

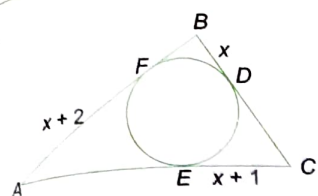
$$\Rightarrow \angle EAT = 90^\circ$$

Hence, $AT = \sqrt{2}$. Now, $\angle ATX = 150^\circ$



Since $TX = 2$, by applying the cosine rule in $\triangle ATX$, we get

$$(AX)^2 = 3 + 4 - 4\sqrt{2} \cos 150^\circ = 7 + 6 = 13$$



Let $BD = x$, $CE = x + 1$ and $AF = x + 2$. Then

$$CD = CE = x + 1 = s - a$$

$$AE = AF = x + 2 = s - b$$

$$BF = BD = x = s - c$$

On adding, we get

$$3s - (a + b + c) = 3x + 3$$

$$\therefore s = 3x + 3$$

$$\text{Now } r = \frac{\Delta}{s} = \frac{1}{s} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{or } 4 = \sqrt{\frac{(x+2)x(x+1)}{3x+3}}$$

$$\text{or } 16 = \frac{x(x+2)}{3}$$

$$\text{or } x^2 + 2x = 48$$

$$\text{or } (x+8)(x-6) = 0$$

$$\text{or } x = 6$$

$$\text{Hence, } s = 21.$$

(211.25)

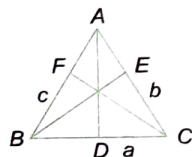
$$\Delta = \frac{1}{2} \times 210a = \frac{1}{2} \times 195b = \frac{1}{2} \times 182c$$

$$\therefore b = \frac{210a}{195} = \frac{14}{13}a; c = \frac{210a}{182} = \frac{15}{13}a$$

$$\text{Hence, } 2s = a + \frac{14a}{13} + \frac{15a}{13} = \frac{(13+14+15)a}{13}$$

$$\text{or } s = \frac{21a}{13}$$

$$\text{Also } \Delta = \sqrt{\frac{21a}{13} \left(\frac{8a}{13} \right) \left(\frac{7a}{13} \right) \left(\frac{6a}{13} \right)} = \frac{84a^2}{169}$$



$$\text{But } \Delta = \frac{1}{2} \times 210a = \frac{84a^2}{169}$$

$$\text{or } a = \frac{105 \times 169}{84} = \frac{15 \times 169}{12} = \frac{845}{4}$$

(35) Using the sine law, we get

$$\frac{27}{\sin \theta} = \frac{48}{\sin 3\theta}, \text{ where } \theta = \angle A$$

$$\Rightarrow \frac{\sin 3\theta}{\sin \theta} = \frac{16}{9}$$

$$\Rightarrow 3 - 4 \sin^2 \theta = \frac{16}{9}$$

$$\Rightarrow 4 \sin^2 \theta = 3 - \frac{16}{9} = \frac{11}{9}$$

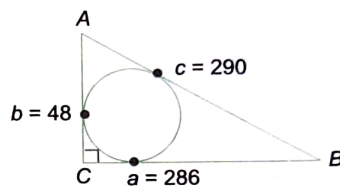
$$\Rightarrow \sin^2 \theta = \frac{11}{36} \quad \dots(i)$$

$$\text{Now } \frac{b}{\sin 4\theta} = \frac{27}{\sin \theta}$$

$$\begin{aligned} \text{or } \frac{b}{27} &= \frac{\sin 4\theta}{\sin \theta} \\ &= \frac{2 \sin 2\theta \cos 2\theta}{\sin \theta} = 4 \cos \theta \cos 2\theta \\ &= 4 \cos \theta (1 - 2 \sin^2 \theta) \end{aligned}$$

$$\begin{aligned} \text{or } \frac{b}{27} &= 4 \sqrt{1 - \sin^2 \theta} (1 - 2 \sin^2 \theta) \\ &= 4 \sqrt{1 - \frac{11}{36}} \left(1 - \frac{22}{36} \right) \\ &= 4 \left(\frac{5}{6} \right) \left(\frac{14}{36} \right) \text{ or } b = 35 \end{aligned}$$

10. (22)



$$\frac{a}{b} = \frac{143}{24} \Rightarrow \frac{a}{143} = \frac{b}{24} = k \text{ (let)}$$

$$\text{Area of triangle } (\Delta) = \frac{1}{2} ab = 6864$$

$$\text{or } 143k \times 24k = 2 \times 6864$$

$$\text{or } 12k^2 \times 143 = 6864$$

$$\text{or } k^2 = 4$$

$$\text{or } k = 2$$

$$\Rightarrow a = 286, b = 48$$

$$\text{Now } c^2 = 48^2 + (286)^2 = 290^2$$

$$\text{or } c = 290$$

Hence, the radius of the circle inscribed in the triangle is given by

$$r = \frac{\Delta}{s} = 22.$$

$$11. (4) \cos A + \sin A - \frac{2}{\cos B + \sin B} = 0$$

$$\text{or } \cos A \cos B + \sin A \sin B + \cos A \sin B + \sin A \cos B = 2$$

$$\text{or } \cos(A - B) + \sin(A + B) = 2$$

$$\text{or } \cos(A - B) = 1 \text{ and } \sin(A + B) = 1$$

$$\Rightarrow A = B, \text{ so } a = b \text{ and } \sin 2A = 1$$

$$\Rightarrow A = 45^\circ$$

$$\text{Hence, } \frac{a+b}{c} = \frac{2a}{a\sqrt{2}} = \sqrt{2}$$

$$12. (8) A + B + C = \pi$$

$$\text{Given } C = 2A$$

$$\Rightarrow B = \pi - 3A$$

$$\text{As } 0 < C < \pi \Rightarrow 0 < 2A < \pi \Rightarrow 0 < A < \frac{\pi}{2}$$

$$\text{By sine rule, } \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\text{or } \frac{a}{\sin A} = \frac{2a}{\sin(\pi - 3A)} \quad \text{or } \frac{a}{1} = \frac{2a}{3 - 4\sin^2 A}$$

$$\text{or } 3 - 4\sin^2 A = 2$$

$$\text{or } \sin^2 A = \frac{1}{4} \quad \text{or } \sin A = \frac{1}{2} \quad \text{or } A = \frac{\pi}{6} \quad \text{or } \frac{5\pi}{6}$$

$$\text{But } 0 < A < \frac{\pi}{2}$$

$$\Rightarrow A = \frac{\pi}{6}, \angle B = \frac{\pi}{2}, \text{ and } \angle C = \frac{\pi}{3}$$

$$\begin{aligned} \Rightarrow a^2 + b^2 + c^2 &= 4R^2 [\sin^2 A + \sin^2 B + \sin^2 C] \\ &= 4R^2 \left[\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{2} + \sin^2 \frac{\pi}{3} \right] \\ &= 4R^2 \left[\frac{1}{4} + 1 + \frac{3}{4} \right] = 8R^2 \end{aligned}$$

$$\text{or } \frac{a^2 + b^2 + c^2}{R^2} = 8$$

3. (-0.125)

$$b(b+c) = a^2$$

$$\therefore bc = a^2 - b^2$$

$$\therefore \sin B \sin C = \sin^2 A - \sin^2 B$$

$$\therefore \sin B \sin C = \sin(A+B) \sin(A-B)$$

$$\Rightarrow \sin B = \sin(A-B)$$

$$\Rightarrow 2B = A$$

$$\text{Similarly, from } c(c+a) = b^2, \text{ we get } 2C = B$$

$$\text{Now, } A+B+C = \pi$$

$$\Rightarrow 2B + B + \frac{B}{2} = \pi$$

$$\Rightarrow B = \frac{2\pi}{7}$$

$$\therefore A = \frac{4\pi}{7}, C = \frac{\pi}{7}$$

$$\begin{aligned} \Rightarrow \cos A \cdot \cos B \cdot \cos C \\ &= \cos \frac{\pi}{7} \cdot \cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \\ &= \frac{1}{2^3 \sin \frac{\pi}{7}} \cdot \sin \frac{8\pi}{7} \\ &= \frac{1}{8 \sin \frac{\pi}{7}} \left(-\sin \frac{\pi}{7} \right) \\ &= -\frac{1}{8} = -0.125 \end{aligned}$$

4. (3) $a+b-c=2$

$$\text{and } 2ab - c^2 = 4$$

$$\Rightarrow a^2 + b^2 + c^2 + 2ab - 2bc - 2ca = 4 = 2ab - c^2$$

$$\Rightarrow (b-c)^2 + (a-c)^2 = 0$$

$$\Rightarrow a = b = c$$

Triangle is equilateral; hence

$$a = 2$$

$$\Rightarrow \Delta = \sqrt{3}$$

5. (6) Given $s-a=5, s-b=3, s-c=2$

$$\Rightarrow s=10; a=5, b=7, \text{ and } c=8$$

Now, Length of tangent from A to incircle

= Semiperimeter of ΔAPQ

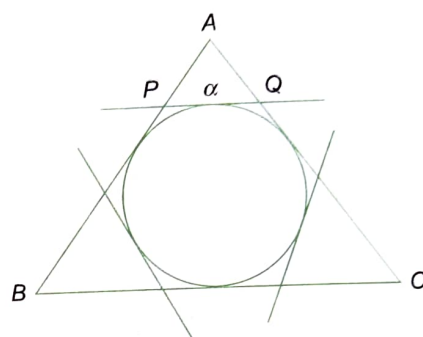
$$5 = \frac{\frac{c\alpha}{a} + \alpha + \frac{b\alpha}{a}}{2}$$

$$\Rightarrow \alpha = \frac{10a}{a+b+c} = \frac{5}{2} \quad [\because \Delta APQ \text{ and } \Delta ABC \text{ are similar}]$$

$$\text{Similarly, } \beta = \frac{6b}{a+b+c} = \frac{21}{10}$$

$$\text{and } \gamma = \frac{4c}{a+b+c} = \frac{8}{5}$$

$$\Rightarrow \alpha + \beta + \gamma = \frac{62}{10}$$



16. (2) $0 < a < b+c$ or $\frac{a}{b+c} < 1$

$$\therefore \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 3$$

$$\text{Also, } [(b+c) + (c+a) + (a+b)]$$

$$\times \left[\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right] \geq 9 \quad (\text{A.M.} \geq \text{H.M.})$$

$$\text{or } (a+b+c) \left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq \frac{9}{2}$$

$$\text{or } \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

$$\text{or } \frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 3$$

Therefore, integral values of

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 2$$

17. (3) The given equation is

$$4 \sin A \sin B + 4 \sin B \sin C + 4 \sin C \sin A = 9$$

$$\Rightarrow 2 \cos(A-B) - 2 \cos(A+B) + 2 \cos(B-C) - 2 \cos(B+C) + 2 \cos(C-A) - 2 \cos(C+A) = 9$$

$$\text{or } 2[\cos(A-B) + \cos(B-C) + \cos(C-A)] = 9 - 2(\cos A + \cos B + \cos C)$$

$$\geq 9 - 2 \times \frac{3}{2} = 6$$

$$\text{or } \cos(A-B) + \cos(B-C) + \cos(C-A) \geq 3$$

$$\text{But } \cos(A-B) \leq 1, \cos(B-C) \leq 1, \cos(C-A) \leq 1$$

$$\text{or } \cos(A-B) = 1, \cos(B-C) = 1, \cos(C-A) = 1$$

Thus, $A=B=C$, i.e., triangle ABC is an equilateral triangle.

$$\text{Hence, } \Delta = \sqrt{3}$$

$AD = h$
 From the figure,
 $\triangle ABD, h^2 + x^2 = (52)^2$
 $\triangle ADC, h^2 + (56 - x)^2 = (60)^2$
 Solving, we get $x = 20$ and $h = 48$
 Then from the property of angle bisector, we
 $BE = y$.

$$\frac{AB}{AC} = \frac{BE}{EC}$$

$$\frac{52}{60} = \frac{y}{56 - y}$$

$$60y = 52 \times 56 - 52y$$

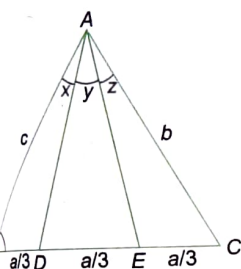
$$112y = 52 \times 56$$

$$y = 26$$

$$DE = BE - BD$$

$$= (26 - 20)$$

$$= 6$$



$\triangle ABD,$

$$\frac{a}{3 \sin x} = \frac{AD}{\sin B} \quad \dots(1)$$

$\triangle ABE,$

$$\frac{2a}{3 \sin(x+y)} = \frac{AE}{\sin B} \quad \dots(2)$$

Dividing Eq. (1) by Eq. (2), we get

$$\frac{\sin(x+y)}{2 \sin x} = \frac{AD}{AE} \quad \dots(3)$$

$\triangle ADC,$

$$\frac{2a}{3 \sin(y+z)} = \frac{AD}{\sin C} \quad \dots(4)$$

$\triangle AEC,$

$$\frac{a}{3 \sin z} = \frac{AE}{\sin C} \quad \dots(5)$$

Dividing Eq. (5) by Eq. (4), we get

$$\frac{\sin(y+z)}{2 \sin z} = \frac{AE}{AD} \quad \dots(6)$$

Multiplying Eq. (3) and Eq. (6), we get

$$\frac{\sin(x+y) \sin(y+z)}{\sin x \cdot \sin z} = 4$$

$$20. (10) \frac{(IA)(IB)(IC)}{(ID)(IE)(IF)} = \frac{\left(r \cos \frac{A}{2}\right) \left(r \cos \frac{B}{2}\right) \left(r \cos \frac{C}{2}\right)}{r^3}$$

$$= \frac{1}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{4R}{r} = 10$$

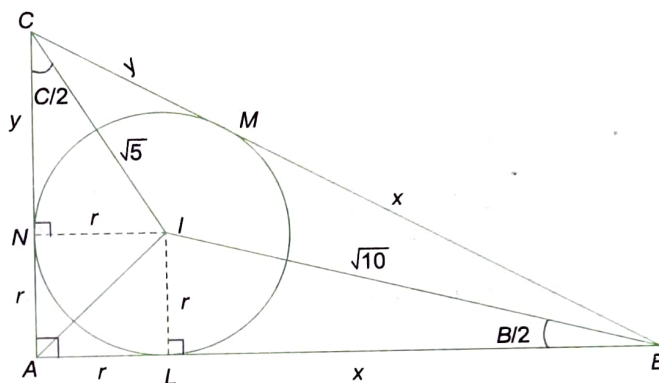
$$21. (4) \text{ Perimeter of } \triangle DEF = a \cos A + b \cos B + c \cos C$$

$$= R [\sin 2A + \sin 2B + \sin 2C]$$

$$= R [4 \sin A \sin B \sin C]$$

$$= 4R \frac{abc}{8R^3} = \frac{abc}{2R^2} = \frac{2\Delta}{R} = 4 \text{ cm}$$

22. (12)



In the figure, $BL = BM = x$

$$CM = CN = y$$

and $AN = AL = r$ (in radius)

$$\text{In } \triangle ILB, \sin \frac{B}{2} = \frac{r}{IB} = \frac{r}{\sqrt{10}}$$

$$\text{In } \triangle INC, \sin \frac{C}{2} = \frac{r}{IC} = \frac{r}{\sqrt{5}}$$

$$\Rightarrow \sin \frac{B}{2} = \frac{1}{\sqrt{2}} \sin \frac{C}{2} \quad \dots(i)$$

$$\text{Also, } \frac{B}{2} = \frac{\pi}{4} - \frac{C}{2}$$

$$\Rightarrow \sin \frac{B}{2} = \frac{1}{\sqrt{2}} \left(\cos \frac{C}{2} - \sin \frac{C}{2} \right) \quad \dots(ii)$$

From (i) and (ii), we get

$$\cos \frac{C}{2} = 2 \sin \frac{C}{2}$$

$$\therefore \tan \frac{C}{2} = \frac{1}{2} = \frac{r}{y} \Rightarrow y = 2r$$

Now,

$$y^2 + r^2 = 5$$

$$\Rightarrow 5r^2 = 5$$

$$\Rightarrow r = 1, y = 2 \text{ and } x = 3$$

$$\therefore AB + BC + CA = 2(r + x + y) = 12$$

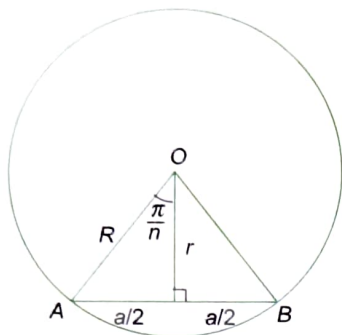
Archives

JEE MAIN

Single Correct Answer Type

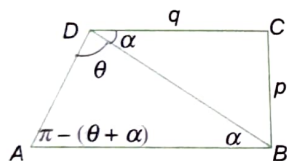
1. (4) $r = \frac{a}{2} \cot \frac{\pi}{n}$ and $R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

$$\therefore \frac{r}{R} = \frac{\cot \frac{\pi}{n}}{\operatorname{cosec} \frac{\pi}{n}} = \cos \frac{\pi}{n}$$



$$\therefore \cos \frac{\pi}{n} \neq \frac{2}{3} \text{ for any } n \in \mathbb{N}$$

2. (1)



$$BD = \sqrt{p^2 + q^2}$$

$$\angle ABD = \angle BDC = \alpha$$

$$\Rightarrow \angle DAB = \pi - (\theta + \alpha)$$

$$\tan \alpha = \frac{p}{q}$$

In $\triangle ABD$, using sine rule

$$\frac{AB}{\sin \theta} = \frac{BD}{\sin (\pi - (\theta + \alpha))} = \frac{BD}{\sin (\theta + \alpha)}$$

$$\begin{aligned} \therefore AB &= \frac{BD \sin \theta}{\sin (\theta + \alpha)} = \frac{BD^2 \sin \theta}{BD \sin \theta \cos \alpha + BD \cos \theta \sin \alpha} \\ &= \frac{BD^2 \sin \theta}{BD \sin \theta \cos \alpha + BD \cos \theta \sin \alpha} \\ &= \frac{(p^2 + q^2) \sin \theta}{q \sin \theta + p \cos \theta} \end{aligned}$$

JEE ADVANCED

Single Correct Answer Type

1. (2) Using cosine rule for $\angle C$, we get

$$\frac{\sqrt{3}}{2} = \frac{(x^2 + x + 1)^2 + (x^2 - 1)^2 - (2x + 1)^2}{2(x^2 + x + 1)(x^2 - 1)}$$

$$\text{or } \sqrt{3} = \frac{2x^2 + 2x - 1}{x^2 + x + 1}$$

$$\text{or } (\sqrt{3} - 2)x^2 + (\sqrt{3} - 2)x + (\sqrt{3} + 1) = 0$$

$$\text{or } x = \frac{(2 - \sqrt{3}) \pm \sqrt{3}}{2(\sqrt{3} - 2)}$$

$$\text{or } x = -(2 + \sqrt{3}), 1 + \sqrt{3}$$

$$\text{or } x = 1 + \sqrt{3} \text{ as } (x > 0).$$

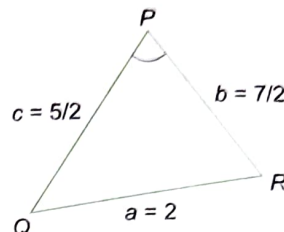
2. (4) Since angles of $\triangle ABC$ are in A.P., $2B = A + C$

$$\text{Also, } A + B + C = 180^\circ$$

$$\therefore B = 60^\circ$$

$$\begin{aligned} \therefore \frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A &= 2 \sin A \cos C + 2 \sin C \cos A \\ &= 2 \sin(A + C) = 2 \sin B = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}. \end{aligned}$$

3. (3)



$$\frac{2 \sin P - 2 \sin P \cos P}{2 \sin P + 2 \sin P \cos P}$$

$$= \frac{1 - \cos P}{1 + \cos P} = \frac{2 \sin^2 \frac{P}{2}}{2 \cos^2 \frac{P}{2}} = \tan^2 \frac{P}{2}$$

$$= \frac{(s - b)(s - c)}{s(s - a)}$$

$$= \frac{((s - b)(s - c))^2}{\Delta^2} = \frac{\left(\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

Multiple Correct Answers Type

1. (2), (3)

$$\cos B + \cos C = 4 \sin^2 \frac{A}{2}$$

$$\Rightarrow 2 \cos \frac{B+C}{2} \cos \frac{B-C}{2} = 4 \sin^2 \frac{A}{2}$$

$$\Rightarrow 2 \cos \frac{B-C}{2} = 4 \sin \frac{A}{2} \quad \left(\because \cos \frac{B+C}{2} = \sin \frac{A}{2} \right)$$

$$\Rightarrow 2 \cos \frac{A}{2} \cos \frac{B-C}{2} = 4 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\Rightarrow 2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} = 2 \sin A$$

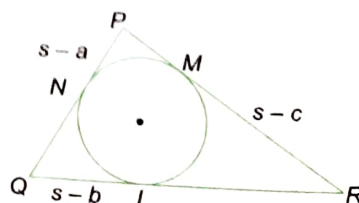
$$\Rightarrow \sin B + \sin C = 2 \sin A$$

$$\Rightarrow b + c = 2a$$

(Using sine rule)

Thus sum of two variable sides b and c is constant $2a$.So locus of vertex A is ellipse with vertices B and C as its foci

2. (2), (4)



Let $s - a = 2k - 2$, $s - b = 2k$, $s - c = 2k + 2$,
 $k \in \mathbb{I}$, $k > 1$

Adding we get,

$$s = 6k. \text{ So, } a = 4k + 2, b = 4k, c = 4k - 2$$

$$\text{Now, } \cos P = \frac{1}{3} \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{3}$$

$$\Rightarrow 3[(4k)^2 + (4k - 2)^2 - (4k + 2)^2] = 2 \times 4k(4k - 2)$$

$$\text{or } 3[16k^2 - 4(4k) \times 2] = 8k(4k - 2)$$

$$\text{or } 48k^2 - 96k = 32k^2 - 16k \text{ or } 16k^2 = 80k \text{ or } k = 5$$

So, sides are 22, 20, 18.

3. (1), (3), (4)

$$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{3s-(x+y+z)}{9} = \frac{s}{9}$$

$$\therefore x = \frac{5s}{9}, y = \frac{2s}{3}, z = \frac{7s}{9}$$

$$\text{Area of incircle} = \pi r^2 = \frac{8\pi}{3} \Rightarrow r^2 = \frac{8}{3}$$

$$\Rightarrow \Delta^2 = r^2 s^2 = \frac{8s^2}{3}$$

$$\Rightarrow s(s-x)(s-y)(s-z) = \frac{8s^2}{3}$$

$$\Rightarrow s \cdot \frac{4s}{9} \cdot \frac{s}{3} \cdot \frac{2s}{9} = \frac{8}{3}s^2$$

$$\Rightarrow s = 9$$

$$\therefore \Delta = \sqrt{\frac{8}{3}} \times 9 = 6\sqrt{6} \text{ sq. units}$$

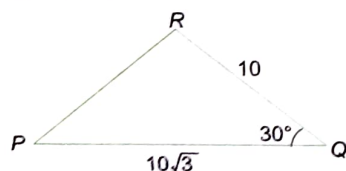
$$R = \frac{xyz}{4\Delta} = \frac{\frac{5s}{9} \cdot \frac{2s}{3} \cdot \frac{7s}{9}}{4 \times 6\sqrt{6}} = \frac{35}{24}\sqrt{6}$$

$$\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{r}{4R} = \frac{\sqrt{\frac{8}{3}}}{4 \cdot \frac{35\sqrt{6}}{24}} = \frac{4}{35}$$

$$\sin^2 \left(\frac{X+Y}{2} \right) = \cos^2 \left(\frac{Z}{2} \right) = \frac{1 + \cos Z}{2} = \frac{3}{5}$$

(Using cosine rule)

4. (2), (3), (4)



$$\cos Q = \frac{100 + 300 - (PR)^2}{2 \times 10 \times 10\sqrt{3}}$$

$$\Rightarrow \cos 30^\circ = \frac{400 - (PR)^2}{200\sqrt{3}}$$

$$\Rightarrow 300 = 400 - (PR)^2$$

$$\Rightarrow PR = 10$$

So, triangle is isosceles.

$$\therefore \angle QPR = 30^\circ \text{ and } \angle PRQ = 120^\circ$$

$$\begin{aligned} \text{Area of triangle, } \Delta &= \frac{1}{2} \times (PQ) \times (QR) \sin Q \\ &= \frac{1}{2} \times 10 \times 10\sqrt{3} \times \frac{1}{2} = 25\sqrt{3} \end{aligned}$$

$$\text{Now, } 2s = 20 + 10\sqrt{3}$$

$$\text{Inradius, } r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{10 + 5\sqrt{3}} = \frac{5\sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{5\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$$

$$= 10\sqrt{3} - 15$$

Using sine rule, we get

$$\frac{PR}{\sin Q} = 2R', \text{ where } R' \text{ is circumradius.}$$

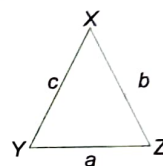
$$\text{or } \frac{10}{\sin 30^\circ} = 2R'$$

$$\therefore R' = 10$$

$$\text{Hence, area of circumcircle} = \pi(R')^2 = 100\pi$$

Matrix Match Type

1. (a) $\rightarrow$ (p), (r), (s)



$$\text{Given } 2(a^2 - b^2) = c^2$$

$$\Rightarrow 2(\sin^2 X - \sin^2 Y) = \sin^2 Z$$

$$\Rightarrow 2 \sin(X+Y) \sin(X-Y) = \sin^2 Z$$

$$\Rightarrow 2 \sin(\pi - Z) \sin(X-Y) = \sin^2 Z$$

$$\Rightarrow \sin(X-Y) = \frac{\sin Z}{2}$$

...(i)

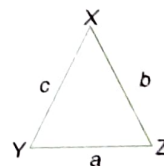
$$\therefore \lambda = \frac{\sin(X-Y)}{\sin Z} = \frac{1}{2}$$

$$\text{Now } \cos(n\pi\lambda) = 0$$

$$\Rightarrow \cos\left(\frac{n\pi}{2}\right) = 0$$

$$\therefore n = 1, 3, 5$$

(b) $\rightarrow$ (p)



$$1 + \cos 2X - 2 \cos 2Y = 2 \sin X \sin Y$$

$$2 \cos^2 X - 2 \cos 2Y = 2 \sin X \sin Y$$

$$1 - \sin^2 X - 1 + 2 \sin^2 Y = \sin X \sin Y$$

$$\sin^2 X + \sin X \sin Y = 2 \sin^2 Y$$

$$\sin X (\sin X + \sin Y) = 2 \sin^2 Y$$

$$\Rightarrow a(a+b) = 2b^2$$

$$\Rightarrow a^2 + ab - 2b^2 = 0$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 2 = 0$$

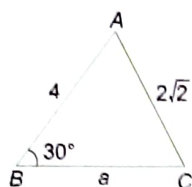
$$\Rightarrow \frac{a}{b} = -2, 1$$

$$\Rightarrow \frac{a}{b} = 1$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (4)



$$\Rightarrow \cos 30^\circ = \frac{a^2 + 16 - 8}{2 \times a \times 4}$$

$$\frac{\sqrt{3}}{2} = \frac{a^2 + 8}{8a}$$

$$\Rightarrow a^2 - 4\sqrt{3}a + 8 = 0$$

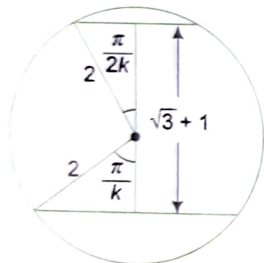
$$\Rightarrow a_1 + a_2 = 4\sqrt{3}, a_1 a_2 = 8$$

$$\Rightarrow |a_1 - a_2| = 4$$

$$\begin{aligned} &= \left| \frac{1}{2} a_1 \times 4 \sin 30^\circ - \frac{1}{2} a_2 \times 4 \sin 30^\circ \right| \\ &= 4 \times \frac{1}{2} \times 4 \sin 30^\circ \end{aligned}$$

$$\Rightarrow |\Delta_1 - \Delta_2| = 4$$

2. (3)



$$2 \cos \frac{\pi}{2k} + 2 \cos \frac{\pi}{k} = \sqrt{3} + 1$$

$$\text{or } \cos \frac{\pi}{2k} + \cos \frac{\pi}{k} = \frac{\sqrt{3} + 1}{2}$$

Let $\frac{\pi}{k} = \theta$. Then,

$$\cos \theta + \cos \frac{\theta}{2} = \frac{\sqrt{3} + 1}{2}$$

$$\text{or } 2 \cos^2 \frac{\theta}{2} - 1 + \cos \frac{\theta}{2} = \frac{\sqrt{3} + 1}{2}$$

$$\text{or } 2t^2 + t - \frac{\sqrt{3} + 3}{2} = 0 \quad [\text{where } \cos(\theta/2) = t]$$

$$t = \frac{-1 \pm \sqrt{1 + 4(3 + \sqrt{3})}}{4}$$

$$\text{or } = \frac{-1 \pm (2\sqrt{3} + 1)}{4} = \frac{-2 - 2\sqrt{3}}{4}, \frac{\sqrt{3}}{2}$$

$$\Rightarrow t = \cos \frac{\theta}{2} \in [-1, 1] \therefore \cos \frac{\theta}{2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \frac{\theta}{2} = \frac{\pi}{6} \Rightarrow k = 3.$$

$$3. (3) \quad \Delta = \frac{1}{2} ab \sin C \Rightarrow \sin C = \frac{2\Delta}{ab} = \frac{2 \times 15\sqrt{3}}{6 \times 10} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow C = 120^\circ$$

$$\Rightarrow c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

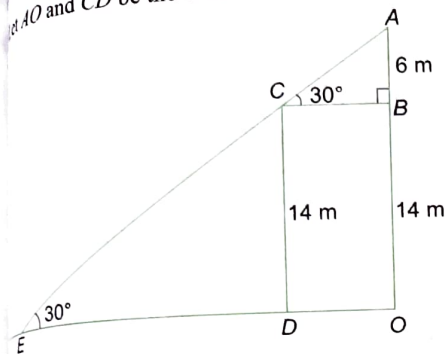
$$= \sqrt{6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ} = 14$$

$$\text{Now } r = \frac{\Delta}{s}$$

$$\Rightarrow r^2 = \frac{225 \times 3}{\left(\frac{6 + 10 + 14}{2}\right)^2} = 3$$

Correct Answer Type

Let AO and CD be the towers.



In triangle ABC ,

$$AC = 6 \operatorname{cosec} 30^\circ = 12 \text{ m}$$

Let OA be the unfinished tower and it is to be raised to point B . Also, let C be the point of observation.

In the figure,

$$\frac{h+x}{120} = \tan 60^\circ = \sqrt{3}$$

$$h+x = 120\sqrt{3}$$

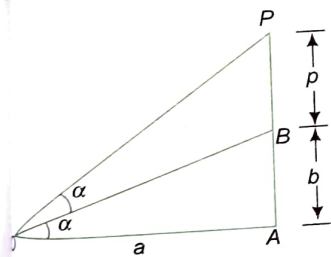
$$\text{Also, } \frac{h}{120} = \tan 45^\circ = 1$$

$$h = 120 \text{ m}$$

$$\Rightarrow 120 + x = 120\sqrt{3}$$

$$\Rightarrow x = 120(\sqrt{3} - 1) \text{ m}$$

Let AB be the tower and PB be the pole.



From the figure, $\tan \alpha = \frac{b}{a}$

$$\text{and } \tan 2\alpha = \frac{p+b}{a}$$

$$\frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{p+b}{a}$$

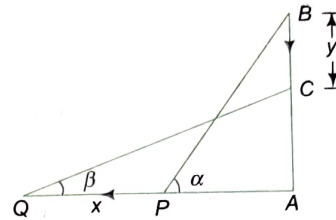
$$\Rightarrow \frac{2\left(\frac{b}{a}\right)}{1 - \left(\frac{b}{a}\right)^2} = \frac{p+b}{a}$$

$$\Rightarrow \frac{2ba}{a^2 - b^2} = \frac{p+b}{a}$$

$$\Rightarrow \frac{2ba^2 - a^2b + b^3}{a^2 - b^2} = p$$

$$\Rightarrow p = b \left(\frac{a^2 + b^2}{a^2 - b^2} \right)$$

4. (2) Let AB be the wall, P be the initial position and Q be the final position of the foot of the ladder on the ground.



In the figure, $PB = QC = l$

(Length of ladder)

In triangle BAP , $PA = l \cos \alpha$, $AB = l \sin \alpha$

In triangle CAQ , $QA = l \cos \beta$, $AC = l \sin \beta$

Now, $y = BC = AB - AC = l(\sin \alpha - \sin \beta)$

and $x = PQ = AQ - AP = l(\cos \beta - \cos \alpha)$

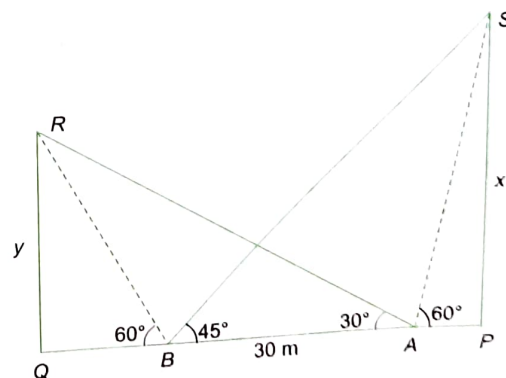
$$\Rightarrow \frac{y}{x} = \frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha}$$

$$= \frac{2 \sin \left(\frac{\alpha - \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right)}{2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)}$$

$$\Rightarrow \frac{y}{x} = \cot \left(\frac{\alpha + \beta}{2} \right)$$

$$\Rightarrow x = y \tan \left(\frac{\alpha + \beta}{2} \right)$$

5. (4) Let x and y be the heights of the flagstaffs at P and Q , respectively.



$$\text{Then } AP = x \cot 60^\circ = \frac{x}{\sqrt{3}},$$

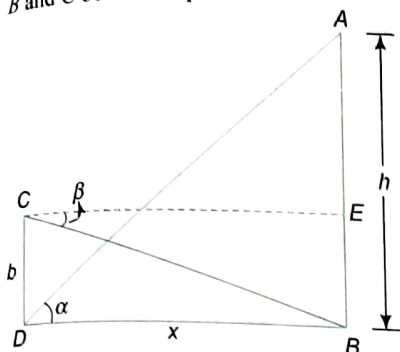
$$AQ = y \cot 30^\circ = y\sqrt{3},$$

$$BP = x \cot 45^\circ = x,$$

$$\text{and } BQ = y \cot 60^\circ = \frac{y}{\sqrt{3}}$$

$$\therefore H = \frac{h \cot(\alpha - \beta)}{\cot(\alpha - \beta) - \cot \alpha}$$

11. (3) Let AB be the tower of height h , D be the point x m away from B and C be another point b m above D .



In triangle ABD ,

$$h = x \tan \alpha \quad \dots (1)$$

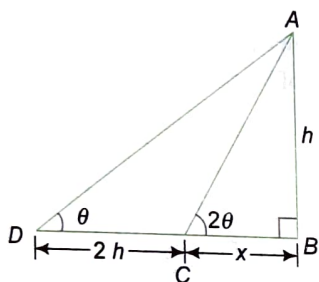
In triangle BCE ,

$$x = b \cot \beta \quad \dots (2)$$

From (1) and (2), we get

$$\therefore h = b \tan \alpha \cot \beta$$

12. (4) Let AB be the pole of height h . Also, let D and C be the initial and final positions of the man.



In triangle ABC ,

$$x = h \cot 2\theta \quad \dots (1)$$

In triangle ABD ,

$$h = (2h + x) \tan \theta \quad \dots (2)$$

Put the value of x from (1) into (2), we get

$$1 = \left(2 + \frac{1}{\tan 2\theta} \right) \tan \theta$$

$$\Rightarrow 1 = \left(2 + \frac{1 - \tan^2 \theta}{2 \tan \theta} \right) \tan \theta$$

$$\therefore \tan^2 \theta - 4 \tan \theta + 1 = 0$$

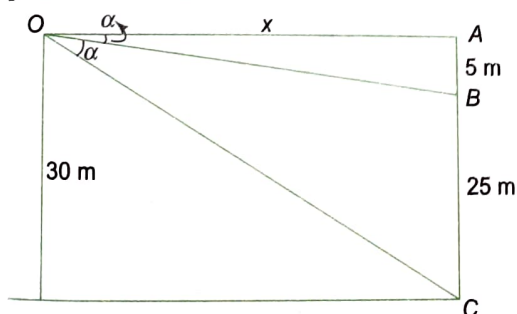
$$\Rightarrow \tan \theta = 2 \pm \sqrt{3}$$

$$\tan \theta = 2 + \sqrt{3} \quad (\text{Rejected as otherwise } 2\theta = 150^\circ)$$

$$\therefore \tan \theta = (2 - \sqrt{3})$$

$$\Rightarrow \theta = 15^\circ$$

13. (3) Let AB be the pole standing on the building BC and O be the position of the antenna.



In triangle OAB ,

$$\tan \alpha = \frac{5}{x} \quad \dots (1)$$

In triangle OAC ,

$$\tan 2\alpha = \frac{30}{x} \quad \dots (2)$$

From (1) and (2),

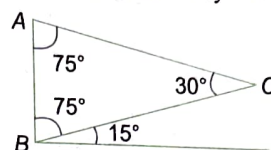
$$\tan 2\alpha = 6 \tan \alpha$$

$$\Rightarrow 3 - 3 \tan^2 \alpha = 1$$

$$\Rightarrow \tan \alpha = \sqrt{\frac{2}{3}}$$

$$\text{From (1), } x = 5 \cot \alpha = 5\sqrt{\frac{3}{2}}$$

14. (3) Let BC be the declivity and BA be the tower.



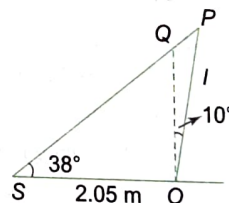
Using sine rule in $\triangle ABC$, we get

$$\frac{BC}{\sin 75^\circ} = \frac{AB}{\sin 30^\circ}$$

$$\Rightarrow AB = \frac{80 \sin 30^\circ}{\sin 75^\circ}$$

$$\Rightarrow AB = \frac{40 \times 2\sqrt{2}}{\sqrt{3} + 1} = 40(\sqrt{6} - \sqrt{2})$$

15. (3) Let PO be the tower and OS be its shadow.



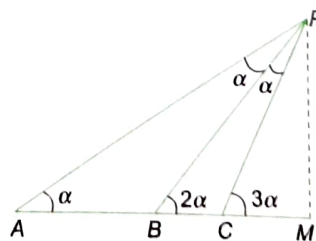
In triangle SOP , using sine rule,

$$\frac{l}{\sin 38^\circ} = \frac{2.05}{\sin(\angle SPO)}$$

$$\angle SPO = 180^\circ - 38^\circ - 90^\circ - 10^\circ = 42^\circ$$

$$\text{So, from (1), } l = \frac{2.05 \sin 38^\circ}{\sin 42^\circ} \quad \dots (1)$$

16. (2) Let PM be the tower.



Given $\angle BAP = \alpha$, $\angle CBP = 2\alpha$ and $\angle MCP = 3\alpha$

$$\therefore \angle APB = \angle PBC - \angle BAP = 2\alpha - \alpha = \alpha$$

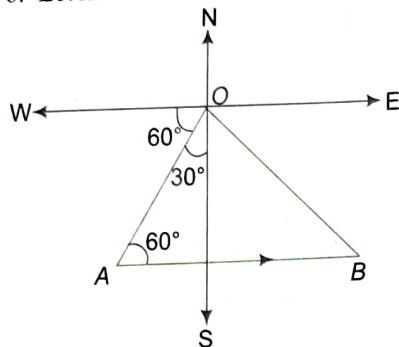
Similarly, $\angle BPC = \alpha$

$$\Rightarrow BP = AB$$

By sine formula in triangle BCP , we have

$$\begin{aligned} \frac{BC}{\sin(\angle BPC)} &= \frac{BP}{\sin(\angle BCP)} \\ \Rightarrow \frac{BC}{\sin \alpha} &= \frac{AB}{\sin(180^\circ - 3\alpha)} \\ \Rightarrow \frac{AB}{BC} &= \frac{\sin 3\alpha}{\sin \alpha} = \frac{3 \sin \alpha - 4 \sin^3 \alpha}{\sin \alpha} \\ &= 3 - 4 \sin^2 \alpha \\ &= 3 - 2(1 - \cos 2\alpha) \\ &= 1 + 2 \cos 2\alpha \end{aligned}$$

17. (2) Let A be the position of the harbour and O be the fort. $OA = 30$, $OB = 70$. Let $AB = x$.

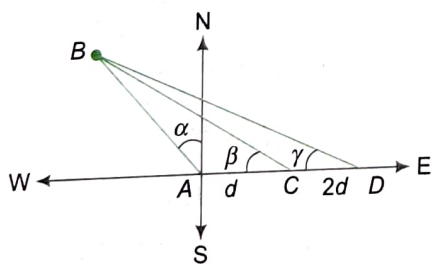


Applying cosine formula in $\triangle OAB$, we get

$$\begin{aligned} \cos 60^\circ &= \frac{OA^2 + AB^2 - OB^2}{2OA \cdot AB} \\ \Rightarrow \frac{1}{2} &= \frac{900 + x^2 - 4900}{2 \times 30 \times x} \\ \Rightarrow x^2 - 30x - 4000 &= 0 \\ \Rightarrow x &= 80 \\ AB &= 80 \text{ km} \end{aligned}$$

Speed of ship = 10 km/hr
 $\therefore$ Time = 8 hr

18. (2) Observe the following figure.



Using $m-n$ theorem in triangle ABD , we have
 $(d + 2d) \cot \beta = d \cot \gamma - 2d \cot(90^\circ + \alpha)$
 $\Rightarrow 3d \cot \beta = d \cot \gamma + 2d \tan \alpha$
 $\Rightarrow 3 \cot \beta = \cot \gamma + 2 \tan \alpha$
 $\therefore 2 \tan \alpha = 3 \cot \beta - \cot \gamma$

Archives

JEE MAIN

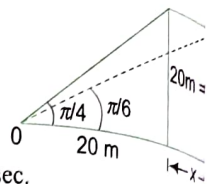
Single Correct Answer Type

1. (4) $\tan 30^\circ = \frac{20}{20+x} = \frac{1}{\sqrt{3}}$

$$\Rightarrow 20 + x = 20\sqrt{3}$$

$$\Rightarrow x = 20(\sqrt{3} - 1)$$

$$\Rightarrow \text{Speed is } 20(\sqrt{3} - 1) \text{ m/sec.}$$



2. (1) Let ED be the tower of height h .

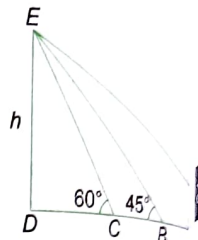
$$\tan 30^\circ = \frac{h}{AD}$$

$$\Rightarrow AD = h\sqrt{3}$$

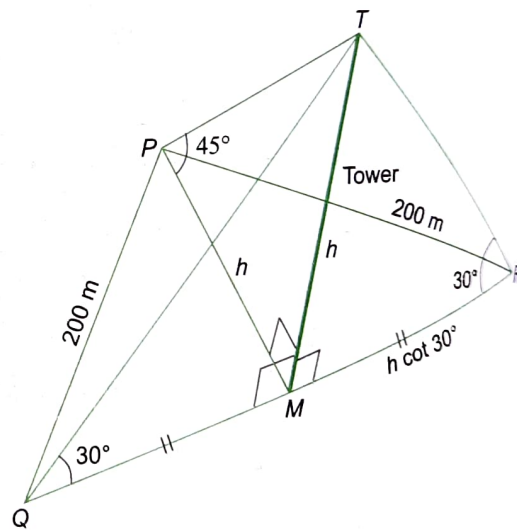
$$BD = h; CD = \frac{h}{\sqrt{3}}$$

$$\frac{AB}{BC} = \frac{AD - BD}{BD - CD}$$

$$= \frac{h\sqrt{3} - h}{h - \frac{h}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\left(\frac{\sqrt{3} - 1}{\sqrt{3}}\right)} = \sqrt{3}$$



3. (2) In the figure PQR is triangular park with $PQ = PR = 200$ m. TV tower of height ' h ' stands at midpoint M of QR .



In triangle TMR , $MR = h \cot 30^\circ$

In triangle TMP , $PM = TM = h$

In triangle PMR , $PR^2 = PM^2 + MR^2$

$$\therefore 200^2 = h^2 + 3h^2$$

$$\therefore h = 100 \text{ m}$$

Concept Application Exercises

Exercise 7.1

1. (a) Let θ be the principal value of $\operatorname{cosec}^{-1}(-1)$.
 $\theta \in [-\pi/2, \pi/2] - \{0\}$ and $\operatorname{cosec}^{-1}(-1) = \theta$

$$\therefore \theta = -\frac{\pi}{2} \text{ because } -\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

$$\text{and } \operatorname{cosec}\left(-\frac{\pi}{2}\right) = -1$$

Therefore, principal value of $\operatorname{cosec}^{-1}(-1)$ is $-\frac{\pi}{2}$.

- (b) Let θ be the principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

$$\therefore \theta \in (0, \pi) \text{ and } \cot \theta = -\frac{1}{\sqrt{3}}$$

$$\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}, \text{ because } \frac{2\pi}{3} \in (0, \pi)$$

$$\text{and } \cot\left(\pi - \frac{\pi}{3}\right) = -\cot \frac{\pi}{3} = -\frac{1}{\sqrt{3}}$$

Therefore, principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{3}$.

2. $\cos^{-1} x < 2$

$$\text{or } 0 \leq \cos^{-1} x < 2$$

$$\text{or } \cos 0 \geq x > \cos 2 \quad (\because \cos^{-1} x \text{ is decreasing function})$$

$$\Rightarrow x \in (\cos 2, 1]$$

3. We have $f(x) = \sin^{-1}(1-x) + \cos^{-1}\sqrt{x-2}$

$$\text{We must have } -1 \leq (1-x) \leq 1$$

$$\Rightarrow -2 \leq -x \leq 0$$

$$\Rightarrow 0 \leq x \leq 2 \text{ and } x-2 \geq 0$$

$$\Rightarrow 0 \leq x \leq 2 \text{ and } x \geq 2$$

$$\therefore x = 2$$

Thus, expression is defined only for $x = 2$.

$$\begin{aligned} \therefore f(2) &= \sin^{-1}(-1) + \cos^{-1}(0) \\ &= (-\pi/2) + (\pi/2) \\ &= 0 \end{aligned}$$

4. We have $f(x) = \cos^{-1}\sqrt{x^2+3x+1} + \cos^{-1}\sqrt{x^2+3x}$

$$\text{We must have } 0 \leq x^2+3x+1 \leq 1 \text{ and } 0 \leq x^2+3x \leq 1$$

$$\Rightarrow x^2+3x = 0$$

$$\Rightarrow x = 0, -3$$

5. $f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right), 0 \leq x \leq 1$

$$\text{Now } \frac{1-x}{1+x} = \frac{2}{1+x} - 1$$

$$\text{Given } 0 \leq x \leq 1,$$

$$\Rightarrow \frac{2}{1+x} - 1 \in [0, 1]$$

$$\Rightarrow \tan^{-1}\left(\frac{1-x}{1+x}\right) \in [\tan^{-1} 0, \tan^{-1} 1] \text{ or } \left[0, \frac{\pi}{4}\right]$$

6. We have

$$\sin^{-1}(\cos^{-1} x) < 1$$

$$\Rightarrow -\frac{\pi}{2} \leq \sin^{-1}(\cos^{-1} x) < 1$$

$$\Rightarrow -1 \leq \cos^{-1} x < \sin 1$$

$$\Rightarrow 0 \leq \cos^{-1} x < \sin 1$$

$$\Rightarrow \cos(\sin 1) < x \leq 1$$

$$\text{Also, } \cos^{-1}(\cos^{-1} x) < 1$$

$$\Rightarrow 0 \leq \cos^{-1}(\cos^{-1} x) < 1$$

$$\Rightarrow \cos 1 < \cos^{-1} x \leq 1$$

$$\Rightarrow \cos 1 \leq x < \cos(\cos 1)$$

$$\text{Now, } 1 > \sin 1$$

$$\Rightarrow \cos 1 < \cos(\sin 1)$$

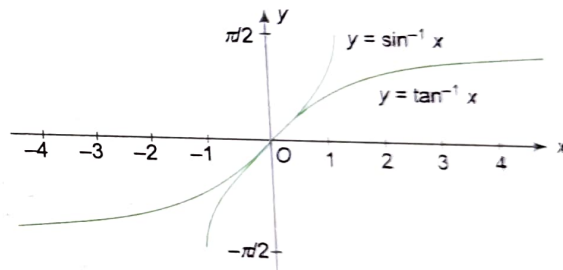
Hence, from (i) and (ii), we have

$$\cos(\sin 1) < x < \cos(\cos 1)$$

7. We have $\sin^{-1} x > \tan^{-1} x$

Given inequality is meaningful if $x \in [-1, 1]$.

Now, let's draw the graphs of $y = \sin^{-1} x$ and $y = \tan^{-1} x$.



From the graph, we can see that $\sin^{-1} x > \tan^{-1} x$ for $x \in (0, 1]$

8. We have $f(x) = \cos^{-1} x + \cot^{-1} x$

Clearly, domain of the function is $[-1, 1]$.

Now, both $\cos^{-1} x$ and $\cot^{-1} x$ are decreasing functions.

$\therefore f(x)$ is also decreasing function

$\therefore$ Range of the function is $[f(1), f(-1)]$

$$\text{Now, } f(-1) = \cos^{-1}(-1) + \cot^{-1}(-1) = \pi + 3\pi/4 = 7\pi/4$$

$$f(1) = \cos^{-1}(1) + \cot^{-1}(1) = 0 + \pi/4 = \pi/4$$

Hence, range is $[\pi/4, 7\pi/4]$.

9. $(\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \frac{3\pi^2}{4}$

$$\Rightarrow \sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \pm \frac{\pi}{2}$$

For minimum value of $x + y + z$, $x = y = z = -1$

$$\text{Hence, } (x + y + z)_{\min} = -3$$

10. Let $\theta = \cos^{-1}\left(\frac{-1}{9}\right)$

$$\Rightarrow \cos \theta = \frac{-1}{9}$$

$$\Rightarrow 2 \cos^2 \frac{\theta}{2} = 1 - \frac{1}{9}$$

$$\Rightarrow \cos \frac{\theta}{2} = \frac{2}{3}$$

$$\Rightarrow 1 - 2 \sin^2 \frac{\theta}{4} = \frac{2}{3}$$

$$\Rightarrow 2 \sin^2 \frac{\theta}{4} = \frac{1}{3}$$

$$\Rightarrow \sin \frac{\theta}{4} = \frac{1}{\sqrt{6}} \quad \left(\text{as } \frac{\pi}{2} < \theta < \pi \Rightarrow \frac{\pi}{8} < \frac{\theta}{4} < \frac{\pi}{4} \right)$$

$$\Rightarrow \sin \left(\frac{1}{4} \cos^{-1} \left(\frac{-1}{9} \right) \right) = \frac{1}{\sqrt{6}}$$

11. Let $x = \cos \theta$

Since $x < 0$, we have $\theta \in [\pi/2, \pi]$

$$\begin{aligned} \text{Now, } \tan^{-1} \frac{\sqrt{1-x^2}}{x} &= \tan^{-1} \frac{\sqrt{1-\cos^2 \theta}}{\cos \theta} \\ &= \tan^{-1} (\tan \theta) \neq \theta \quad [\because \theta \notin (-\pi/2, \pi/2)] \\ &= \tan^{-1} \tan (\pi + (\theta - \pi)) \\ &= \tan^{-1} (\tan (\theta - \pi)) \\ &= \theta - \pi \end{aligned}$$

$$\Rightarrow \cos^{-1} x = \pi + \tan^{-1} \frac{\sqrt{1-x^2}}{x}$$

12. Let $x = \sin \theta$, where

$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

$$\Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$\begin{aligned} \sin^{-1} \left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right) &= \sin^{-1} \left(\frac{\sin \theta + \cos \theta}{\sqrt{2}} \right) \\ &= \sin^{-1} \left(\sin \left(\theta + \frac{\pi}{4} \right) \right) \\ &= \theta + \frac{\pi}{4} \quad \left(\because \theta + \frac{\pi}{4} \in \left(0, \frac{\pi}{2} \right) \right) \\ &= \sin^{-1} x + \frac{\pi}{4} \end{aligned}$$

Exercise 7.2

$$\begin{aligned} 1. \text{ (a) } \tan^{-1} \tan \frac{13\pi}{5} &= \tan^{-1} \tan \left\{ 3\pi + \left(-\frac{2\pi}{5} \right) \right\} \\ &= \tan^{-1} \tan \left(-\frac{2\pi}{5} \right) \\ &= -\frac{2\pi}{5} \end{aligned}$$

$$\begin{aligned} \text{(b) } \sec^{-1} \sec \frac{13\pi}{3} &= \sec^{-1} \sec \left(4\pi + \frac{\pi}{3} \right) \\ &= \sec^{-1} \sec \frac{\pi}{3} \\ &= \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{(c) } \sin^{-1} \sin \frac{33\pi}{5} &= \sin^{-1} \sin \left(7\pi - \frac{2\pi}{5} \right) \\ &= \sin^{-1} \sin \frac{2\pi}{5} = \frac{2\pi}{5} \end{aligned}$$

$$\text{(d) } \sin^{-1} (\sin 8) = \sin^{-1} (\sin (3\pi - 8)) = 3\pi - 8$$

$$\text{(e) } \tan^{-1} (\tan 10) = \tan^{-1} (\tan (10 - 3\pi)) = 10 - 3\pi$$

$$\text{(f) } \sec^{-1} (\sec 9) = \sec^{-1} (\sec (9 - 2\pi)) = 9 - 2\pi$$

$$\text{(g) } \cot^{-1} (\cot 6) = \cot^{-1} (\cot (6 - \pi)) = 6 - \pi$$

$$\text{(h) } \operatorname{cosec}^{-1} (\operatorname{cosec} 7) = \operatorname{cosec}^{-1} (\operatorname{cosec} (7 - 2\pi)) = 7 - 2\pi$$

$$2. f(300) = \sin^{-1} (\sin (\log_2 300))$$

$$\log_2 256 = 8 \text{ and } \log_2 512 = 9$$

$$\therefore (\log_2 300) \in (8, 9)$$

$$\begin{aligned} \therefore \sin^{-1} (\sin (\log_2 300)) &= \sin^{-1} (\sin (3\pi - \log_2 300)) \\ &= 3\pi - \log_2 300 \end{aligned}$$

$$3. f(x) = (\sin^{-1} (\sin x))^2 - \sin^{-1} (\sin x)$$

$$= \left(\sin^{-1} (\sin x) - \frac{1}{2} \right)^2 - \frac{1}{4}$$

$$\text{For maximum value of } f(x), \sin^{-1} (\sin x) = \frac{-\pi}{2}$$

$$\therefore f_{\max} = \left(-\frac{\pi}{2} - \frac{1}{2} \right)^2 - \frac{1}{4} = \frac{\pi}{4} (\pi + 2)$$

$$4. \text{ We have } \sin^{-1} (\sin 5) > x^2 - 4x$$

$$\text{Since, } \frac{3\pi}{2} < 5 < \frac{5\pi}{2}$$

$$\therefore \sin^{-1} (\sin 5) = 5 - 2\pi$$

$$\Rightarrow 5 - 2\pi > x^2 - 4x$$

$$\Rightarrow x^2 - 4x + 4 < 9 - 2\pi$$

$$\Rightarrow (x - 2)^2 < 9 - 2\pi$$

$$\Rightarrow -\sqrt{9 - 2\pi} < x - 2 < \sqrt{9 - 2\pi}$$

$$\Rightarrow 2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$$

$$5. f(x) = \sin^{-1} (\sin x) + \cos^{-1} (\cos x)$$

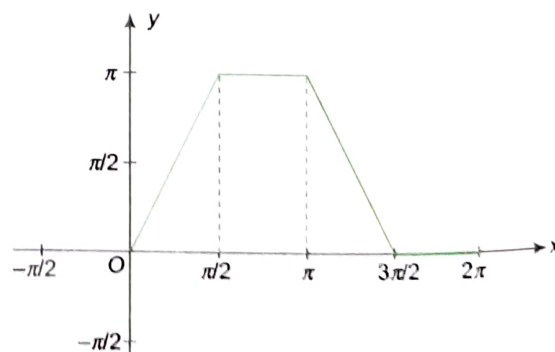
$$= x + x = 2x, \quad x \in [0, \pi/2]$$

$$= \pi - x + x = \pi, \quad x \in [\pi/2, \pi]$$

$$= \pi - x + 2\pi - x = 3\pi - 2x, \quad x \in [\pi, 3\pi/2]$$

$$= x - 2\pi + 2\pi - x = 0, \quad x \in [3\pi/2, 2\pi]$$

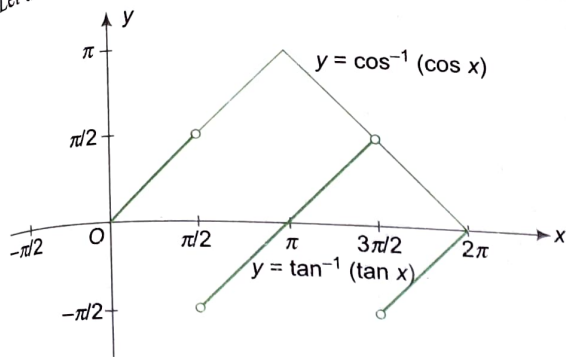
Graph of the function is as shown in the following figure.



From the graph, range is $[0, \pi]$.

Also, area bounded by $y = f(x)$ and x -axis is,

$$\begin{aligned} A &= \frac{1}{2} \left(\frac{3\pi}{2} + \frac{\pi}{2} \right) (\pi) \\ &= \pi^2 \text{ sq. units} \end{aligned}$$



In the above figure, graph of $y = f(x)$ and $y = g(x)$ coincide for $x \in [0, \pi/2) \cup \{2\pi\}$.

Exercise 7.3

1. Putting $x = a \tan^2 \theta$

$$\begin{aligned} \therefore \sin^{-1} \frac{\sqrt{x}}{\sqrt{x+a}} &= \sin^{-1} \frac{\sqrt{a} \sqrt{\tan^2 \theta}}{\sqrt{a \tan^2 \theta + a}} \\ &= \sin^{-1} \frac{\sqrt{a} \tan \theta}{\sqrt{a} \sec \theta} \\ &= \sin^{-1} \sin \theta = \theta = \tan^{-1} \left(\sqrt{\frac{x}{a}} \right) \end{aligned}$$

$$2. \tan(\cos^{-1} x) = \sin\left(\cot^{-1} \frac{1}{2}\right)$$

$$\Rightarrow \tan\left(\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right) = \sin\left(\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$\text{or } \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}}$$

$$\text{or } 4x^2 = 5 - 5x^2$$

$$\text{or } x = \frac{\sqrt{5}}{3} \quad \left[\because x = \frac{-\sqrt{5}}{3}, \text{ L.H.S.} < 0 \text{ but R.H.S.} > 0 \right]$$

$$3. \text{ Here } x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a))))))$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$$

$$= \sqrt{3-a^2}$$

(i)

4. We have to prove that

$$\sin \cot^{-1} \tan \cos^{-1} x = \sin \operatorname{cosec}^{-1} \cot \tan^{-1} x = x$$

$$\text{Now, } \sin \cot^{-1} \tan \cos^{-1} x = \sin \cot^{-1} \tan \tan^{-1} \frac{\sqrt{1-x^2}}{x}$$

$$= \sin \cot^{-1} \frac{\sqrt{1-x^2}}{x} = \sin \sin^{-1} x = x$$

$$\begin{aligned} \text{Also, } \sin \operatorname{cosec}^{-1} \cot \tan^{-1} x &= \sin \operatorname{cosec}^{-1} \frac{1}{x} \\ &= \sin(\sin^{-1} x) = x \end{aligned}$$

5. Let $ax = \tan \theta$

$$\begin{aligned} \Rightarrow \tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right) &= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) \\ &= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) \\ &= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) \\ &= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} \\ &= \frac{\tan^{-1} ax}{2} \end{aligned}$$

6. Let $x = \cos \theta$, where $\theta \in [0, \pi]$

$$\begin{aligned} \Rightarrow y &= \sin\left[2 \tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right] \\ &= \sin\left[2 \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}\right] \\ &= \sin\left[2 \tan^{-1}\left\{\sqrt{\tan^2\frac{\theta}{2}}\right\}\right] \\ &= \sin\left[2 \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}\right] \\ &= \sin \theta \\ &= \sqrt{1-x^2} \end{aligned}$$

$$7. \tan^{-1} \frac{1}{\sqrt{x^2-1}}$$

Let $x = \sec \theta$. Then $\theta = \sec^{-1} x$

$$\text{We have } \tan^{-1} \frac{1}{\sqrt{x^2-1}} = \tan^{-1} \frac{1}{\sqrt{\sec^2\theta-1}}$$

$$= \tan^{-1} \frac{1}{\sqrt{\sec^2\theta-1}}$$

$$= \tan^{-1} \frac{1}{\sqrt{\tan^2\theta}}$$

$$= \tan^{-1} \frac{1}{\tan\theta}$$

($\because x > 1$)

$$= \tan^{-1} \cot \theta$$

$$= \tan^{-1} \tan\left(\frac{\pi}{2} - \theta\right)$$

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1} x$$

8. Put $x = \cos 2\theta$ so that $\theta = \frac{1}{2} \cos^{-1} x$

$$\text{and } 2\theta \in [0, \pi] \text{ or } \theta \in \left[0, \frac{\pi}{2}\right]$$

Then, we have

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$\begin{aligned}
&= \tan^{-1} \left(\frac{\sqrt{2 \cos^2 \theta} - \sqrt{2 \sin^2 \theta}}{\sqrt{2 \cos^2 \theta} + \sqrt{2 \sin^2 \theta}} \right) \\
&= \tan^{-1} \left(\frac{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta} \right) \quad \left(\because \theta \in \left[0, \frac{\pi}{2} \right] \right) \\
&= \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) \\
&= \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \\
&= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \theta \right) \right] \quad \left(\because \frac{\pi}{4} - \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right] \right) \\
&= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x = \text{R.H.S.}
\end{aligned}$$

$$9. f(x) = \cos^{-1} \left(\frac{1+x}{\sqrt{2(1+x^2)}} \right), x \leq 0$$

Putting $x = \tan \theta$, where $\theta \in \left(-\frac{\pi}{2}, 0 \right]$, we get

$$\begin{aligned}
f(x) &= \cos^{-1} \left(\frac{1 + \tan \theta}{\sqrt{2 \sec^2 \theta}} \right) \\
&= \cos^{-1} \left(\frac{1}{\sqrt{2}} (\cos \theta + \sin \theta) \right) \\
&= \cos^{-1} \left(\cos \left(\theta - \frac{\pi}{4} \right) \right)
\end{aligned}$$

$$\text{Now, } \theta - \frac{\pi}{4} \in \left(-\frac{3\pi}{4}, -\frac{\pi}{4} \right]$$

$$\Rightarrow f(x) = -\left(\theta - \frac{\pi}{4} \right) = \frac{\pi}{4} - \tan^{-1} x$$

$$\begin{aligned}
10. \tan^{-1} \left(-\tan \frac{13\pi}{8} \right) + \cot^{-1} \left(-\cot \left(\frac{19\pi}{8} \right) \right) \\
&= -\tan^{-1} \left(\tan \frac{13\pi}{8} \right) + \pi - \cot^{-1} \left(\cot \left(\frac{19\pi}{8} \right) \right) \\
&= -\left(\frac{13\pi}{8} - 2\pi \right) + \pi - \left(\frac{19\pi}{8} - 2\pi \right) \\
&= \frac{3\pi}{8} + \frac{5\pi}{8} = \pi
\end{aligned}$$

$$\begin{aligned}
11. \tan \left\{ \cos^{-1} \left(-\frac{2}{7} \right) - \frac{\pi}{2} \right\} \\
&= \tan \left\{ \pi - \cos^{-1} \left(\frac{2}{7} \right) - \frac{\pi}{2} \right\} \\
&= \tan \left\{ \frac{\pi}{2} - \cos^{-1} \left(\frac{2}{7} \right) \right\} \\
&= \tan \left\{ \sin^{-1} \left(\frac{2}{7} \right) \right\} \\
&= \tan \tan^{-1} \left(\frac{2}{3\sqrt{5}} \right) = \frac{2}{3\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
12. \tan^{-1} \left(\frac{1}{y} \right) &= -\pi + \cot^{-1} y \\
\Rightarrow y &< 0 \\
\Rightarrow x^2 - 3x + 2 &< 0 \\
\Rightarrow (x-1)(x-2) &< 0 \\
\Rightarrow x &\in (1, 2)
\end{aligned}$$

Exercise 7.4

$$\begin{aligned}
1. \sin^{-1} x + \sin^{-1} y &= \frac{2\pi}{3} \\
\Rightarrow \frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \cos^{-1} y &= \frac{2\pi}{3} \\
\Rightarrow \cos^{-1} x + \cos^{-1} y &= \pi - \frac{2\pi}{3} = \frac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
2. \cot^{-1} x + \tan^{-1} 3 &= \frac{\pi}{2} \\
\Rightarrow \cot^{-1} x &= \frac{\pi}{2} - \tan^{-1} 3 \\
\Rightarrow \cot^{-1} x &= \cot^{-1} 3 \\
\therefore x &= 3
\end{aligned}$$

$$\begin{aligned}
3. 2 \cos^{-1} x + \sin^{-1} x &= \frac{2\pi}{3} \\
\Rightarrow \cos^{-1} x + (\cos^{-1} x + \sin^{-1} x) &= \frac{2\pi}{3} \\
\Rightarrow \cos^{-1} x + \frac{\pi}{2} &= \frac{2\pi}{3} \\
\Rightarrow \cos^{-1} x &= \frac{\pi}{6} \Rightarrow x = \frac{\sqrt{3}}{2}
\end{aligned}$$

4. Case I: $0 < x < 1$

$$\begin{aligned}
\text{LHS} &= \sin^{-1} (\cos (\sin^{-1} x)) + \cos^{-1} (\sin (\cos^{-1} x)) \\
&= \sin^{-1} (\cos (\cos^{-1} (\sqrt{1-x^2}))) + \cos^{-1} (\sin (\sin^{-1} (\sqrt{1-x^2}))) \\
&= \sin^{-1} (\sqrt{1-x^2}) + \cos^{-1} (\sqrt{1-x^2}) \\
&= \pi/2
\end{aligned}$$

Case II: $-1 < x < 0$

Put $x = -y$, where $y > 0$.

$$\begin{aligned}
\text{LHS} &= \sin^{-1} (\cos (\sin^{-1} (-y))) + \cos^{-1} (\sin (\cos^{-1} (-y))) \\
&= \sin^{-1} (\cos (-\sin^{-1} y)) + \cos^{-1} (\sin (\pi - \cos^{-1} y)) \\
&= \sin^{-1} (\cos (\sin^{-1} y)) + \cos^{-1} (\sin (\cos^{-1} y)) \\
&= \pi/2 \quad (\text{From Case I})
\end{aligned}$$

5. Given equations are

$$\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3} \quad \dots(i)$$

$$\cos^{-1} x - \cos^{-1} y = \frac{\pi}{3} \quad \dots(ii)$$

$$\Rightarrow \left(\frac{\pi}{2} - \sin^{-1} x \right) - \left(\frac{\pi}{2} - \sin^{-1} y \right) = \frac{\pi}{3}$$

Let $\sin^{-1} x = A$

$\sin^{-1} y = B$

Then Eqs. (i) and (ii) become

$$A + B = \frac{2\pi}{3} \quad \dots(iii)$$

$$A - B = -\frac{\pi}{3}$$

(iv)

Solving Eqs. (iii) and (iv), we get

$$A = \frac{\pi}{6}, B = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1} x = \frac{\pi}{6}; \sin^{-1} y = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{1}{2} \text{ and } y = 1$$

$$(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow (\tan^{-1} x + \cot^{-1} x)^2 - 2 \tan^{-1} x \left(\frac{\pi}{2} - \tan^{-1} x \right) = \frac{5\pi^2}{8}$$

$$\Rightarrow \frac{\pi^2}{4} - 2 \times \frac{\pi}{2} \tan^{-1} x + 2(\tan^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1} x)^2 - \pi \tan^{-1} x - \frac{3\pi^2}{8} = 0$$

$$\Rightarrow \tan^{-1} x = -\frac{\pi}{4}, \frac{3\pi}{4}$$

$$\Rightarrow \tan^{-1} x = -\frac{\pi}{4} \quad \left(\because \tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right)$$

$$\therefore x = -1$$

$$7. \sec^{-1} x > \pi/2 - \sec^{-1} x$$

$$\Rightarrow \sec^{-1} x > \pi/4$$

$$\Rightarrow \sec^{-1} x \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right]$$

$$\Rightarrow x \in (-\infty, -1] \cup (\sqrt{2}, \infty)$$

$$8. \tan^{-1} x > \cot^{-1} x$$

$$\Rightarrow \tan^{-1} x > \pi/2 - \tan^{-1} x$$

$$\Rightarrow \tan^{-1} x > \pi/4$$

$$\Rightarrow x > 1$$

$$9. \text{ Let } f(x) = x^3 + bx^2 + cx + 1$$

$$\text{So, } f(0) = 1 > 0,$$

$$f(-1) = b - c < 0$$

$$\text{So, } -1 < \alpha < 0,$$

$$\therefore \tan^{-1}(\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right) = -\frac{\pi}{2}$$

$$10. \tan^{-1}(\cot \alpha) - \cot^{-1}(\tan \alpha)$$

$$= \tan^{-1}\left(\frac{1}{\tan \alpha}\right) - \left(\frac{\pi}{2} - \tan^{-1}(\tan \alpha)\right)$$

$$= -\frac{\pi}{2} + \left(\tan^{-1}\left(\frac{1}{\tan \alpha}\right) + \tan^{-1}(\tan \alpha)\right)$$

$$= -\frac{\pi}{2} - \frac{\pi}{2}$$

$$= -\pi$$

$$11. \text{ For } x \geq 1, \operatorname{cosec}^{-1} x > 0$$

Using A.M. $\geq$ G.M., we have

$$\frac{\sec^{-1} x + \operatorname{cosec}^{-1} x}{2} \geq \sqrt{\sec^{-1} x \operatorname{cosec}^{-1} x}$$

$$\Rightarrow \frac{(\pi/2)}{2} \geq \sqrt{\sec^{-1} x \operatorname{cosec}^{-1} x}$$

$$\Rightarrow \frac{\pi}{4} \geq \sqrt{\sec^{-1} x \operatorname{cosec}^{-1} x}$$

$$\therefore \sec^{-1} x \operatorname{cosec}^{-1} x \leq \frac{\pi^2}{16}$$

Thus maximum value of $\sec^{-1} x \operatorname{cosec}^{-1} x$ is $\frac{\pi^2}{16}$.

12. We must have

$$4 \sin^2 \theta + \sin \theta = 6 \sin \theta - 1$$

$$\Rightarrow 4 \sin^2 \theta - 5 \sin \theta + 1 = 0$$

$$\Rightarrow (4 \sin \theta - 1)(\sin \theta - 1) = 0$$

$$\Rightarrow \sin \theta = \frac{1}{4} \text{ or } \sin \theta = 1$$

 $\sin \theta = 1$ is not possible as otherwise

$$6 \sin \theta - 1 = 4 \sin^2 \theta + \sin \theta = 5$$

$$\therefore \sin \theta = \frac{1}{4}$$

For 10 solutions, $\theta \in [0, 9\pi]$ or $[0, 10\pi]$.Thus, the least value of n is 9.

Exercise 7.5

$$\begin{aligned} 1. \sin^{-1} \frac{3}{5} + \tan^{-1} \frac{1}{7} &= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} \\ &= \tan^{-1} \left(\frac{3/4 + 1/7}{1 - 3/4 \times 1/7} \right) \\ &= \tan^{-1} \left(\frac{25}{25} \right) = \tan^{-1} 1 = \frac{\pi}{4} \end{aligned}$$

$$2. \text{ Since } \frac{x}{y} \times \frac{x+y}{x-y} > 1, \text{ then the expression is equal to}$$

$$\pi + \tan^{-1} \left[\frac{\frac{x}{y} + \frac{x+y}{x-y}}{1 - \frac{x}{y} \times \frac{x+y}{x-y}} \right] = \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$3. \tan^{-1} \frac{1}{\sqrt{2}} + \sin^{-1} \frac{1}{\sqrt{5}} - \cos^{-1} \frac{1}{\sqrt{10}}$$

$$= \tan^{-1} \frac{1}{\sqrt{2}} + \tan^{-1} \frac{1}{2} - \tan^{-1} 3$$

$$= \tan^{-1} \frac{1}{\sqrt{2}} - \left[\tan^{-1} 3 - \tan^{-1} \frac{1}{2} \right]$$

$$= \tan^{-1} \frac{1}{\sqrt{2}} - \left[\tan^{-1} \frac{3 - \frac{1}{2}}{1 + \frac{3}{2}} \right]$$

$$= \tan^{-1} \frac{1}{\sqrt{2}} - \tan^{-1} 1$$

$$= \tan^{-1} \frac{1}{\sqrt{2}} - \tan^{-1} \frac{1}{1 + \frac{1}{\sqrt{2}}}$$

$$= \tan^{-1} \frac{1 - \sqrt{2}}{1 + \sqrt{2}}$$

$$= -\pi + \cot^{-1} \left(\frac{1 + \sqrt{2}}{1 - \sqrt{2}} \right)$$

$$\left(\because \frac{1 - \sqrt{2}}{1 + \sqrt{2}} < 0 \right)$$

$$4. \tan^{-1} \frac{x-1}{x+2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{\frac{x-1}{x+2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x+2} \right) \left(\frac{x+1}{x+2} \right)} \right] = \frac{\pi}{4}$$

$$\Rightarrow \left[\frac{2x(x+2)}{x^2 + 4 + 4x - x^2 + 1} \right] = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{2x(x+2)}{4x+5} = 1$$

$$\Rightarrow 2x^2 + 4x = 4x + 5$$

$$\therefore x = \pm \sqrt{\frac{5}{2}}$$

But for $x = -\sqrt{5/2}$, L.H.S. is negative.

Hence, $x = \sqrt{5/2}$.

$$5. \text{ Given equation is } \tan^{-1} \left(\frac{x}{1-x^2} \right) + \tan^{-1} \left(\frac{1}{x^3} \right) = \frac{3\pi}{4}.$$

Clearly $x \neq \pm 1$

$$\begin{aligned} \tan^{-1} \left(\frac{x}{1-x^2} \right) + \tan^{-1} \left(\frac{1}{x^3} \right) &= \tan^{-1} \frac{\frac{x}{1-x^2} + \frac{1}{x^3}}{1 - \frac{x}{x^3(1-x^2)}} \\ &= \tan^{-1} \frac{x^4 + 1 - x^2}{(x^2 - x^4 - 1)x} \\ &= \tan^{-1} \left(-\frac{1}{x} \right) \end{aligned}$$

$$\therefore \tan^{-1} \left(-\frac{1}{x} \right) = \frac{3\pi}{4}$$

$$\Rightarrow x = 1, \text{ but } x \neq 1$$

So the given equation has no solution.

$$6. 2 \sin^{-1} \frac{3}{5} = 2 \tan^{-1} \frac{3}{4}$$

$$= \tan^{-1} \frac{2 \left(\frac{3}{4} \right)}{1 - \left(\frac{3}{4} \right)^2} = \tan^{-1} \frac{2(3)(4)}{4^2 - 3^2} = \tan^{-1} \frac{24}{7}$$

$$7. \tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right)$$

$$\text{Put } x = a \tan \theta \text{ or } \frac{x}{a} = \tan \theta \text{ or } \theta = \tan^{-1} \frac{x}{a}$$

We have

$$\begin{aligned} \therefore \tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right) &= \tan^{-1} \left(\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta} \right) \\ &= \tan^{-1} \left(\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta} \right) \\ &= \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) \\ &= \tan^{-1} (\tan 3\theta) = 3\theta \\ &= 3 \tan^{-1} \frac{x}{a} \end{aligned}$$

$$8. 2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$\Rightarrow \frac{2 \cos x}{1 - \cos^2 x} = 2 \operatorname{cosec} x$$

$$\Rightarrow \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

$$9. \tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \tan^{-1} x, \quad x > 0$$

If $x \in (0, 1)$

$$\tan^{-1} \left(\frac{1-x}{1+x} \right) = \tan^{-1} 1 - \tan^{-1} x$$

Therefore, given equation reduces to

$$\tan^{-1} 1 - \tan^{-1} x = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \frac{\pi}{4} = \frac{3}{2} \tan^{-1} x$$

$$\Rightarrow \frac{\pi}{6} = \tan^{-1} x$$

$$\therefore x = \frac{1}{\sqrt{3}}$$

For $x \in (1, \infty)$

$$\tan^{-1} \left(\frac{1-x}{1+x} \right) = \pi + \tan^{-1} 1 - \tan^{-1} x = \frac{5\pi}{4} - \tan^{-1} x$$

Therefore, given equation reduces to

$$\frac{5\pi}{4} - \tan^{-1} x = \frac{1}{2} \tan^{-1} x$$

$$\text{or } \frac{5\pi}{6} = \tan^{-1} x, \text{ which is not possible,}$$

$$\text{Hence, only one solution is } x = \frac{1}{\sqrt{3}}.$$

Alternate Solution:

$$\tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow 2 \tan^{-1} \left(\frac{1-x}{1+x} \right) = \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \frac{2 \left(\frac{1-x}{1+x} \right)}{1 - \left(\frac{1-x}{1+x} \right)^2} = \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \frac{2(1-x^2)}{4x} = \tan^{-1} x$$

$$\Rightarrow 1 - x^2 = 2x^2$$

$$\Rightarrow 3x^2 = 1$$

Exercise 7.6

$$x = \pm \frac{1}{\sqrt{3}}$$

But for $x = -\frac{1}{\sqrt{3}}$, L.H.S. of (1) is > 0 , and R.H.S. is < 0

$$xy = 1 + \frac{x}{z} + \frac{y}{z} > 1$$

$$\begin{aligned} \Rightarrow \tan^{-1} x + \tan^{-1} y + \tan^{-1} z \\ = \pi + \tan^{-1} \left[\frac{x+y}{1-xy} \right] + \tan^{-1} z \\ = \pi + \tan^{-1} \left[\frac{xyz-z}{1-xy} \right] + \tan^{-1} z \\ = \pi + \tan^{-1} \left[\frac{z(xy-1)}{1-xy} \right] + \tan^{-1} z \\ = \pi + \tan^{-1}(-z) + \tan^{-1} z = \pi \end{aligned}$$

11. We have $\alpha\beta = -1$ and $\alpha > \beta$.

$$\begin{aligned} \Rightarrow \tan^{-1} \alpha > \tan^{-1} \beta \\ \therefore \tan^{-1} \alpha - \tan^{-1} \beta = \tan^{-1} \frac{\alpha - \beta}{1 + \alpha\beta} \\ = \tan^{-1} \infty = \frac{\pi}{2} \end{aligned}$$

12. Let t_n denotes the n th term of the series. Then

$$\begin{aligned} t_n &= \cot^{-1} 2n^2 \\ &= \tan^{-1} \frac{1}{2n^2} \\ &= \tan^{-1} \frac{(2n+1) - (2n-1)}{1 + (4n^2 - 1)} \\ &= \tan^{-1} (2n+1) - \tan^{-1} (2n-1) \end{aligned}$$

Putting $n = 1, 2, 3, \dots$, etc. in (i), we get

$$\begin{aligned} t_1 &= \tan^{-1} 3 - \tan^{-1} 1 \\ t_2 &= \tan^{-1} 5 - \tan^{-1} 3 \\ t_3 &= \tan^{-1} 7 - \tan^{-1} 5 \\ &\vdots \end{aligned}$$

$$t_n = \tan^{-1} (2n+1) - \tan^{-1} (2n-1)$$

Adding, we get

$$S_n = \tan^{-1} (2n+1) - \tan^{-1} 1$$

as $n \rightarrow \infty, \tan^{-1} (2n+1) \rightarrow \pi/2$

Hence, the required sum is $\frac{\pi}{4}$.

$$\begin{aligned} 13. \sum_{r=1}^n \tan^{-1} \left(\frac{2^{r-1}}{1+2^{2r-1}} \right) &= \sum_{r=1}^n \tan^{-1} \left(\frac{2^{r-1}}{1+2^r \cdot 2^{r-1}} \right) \\ &= \sum_{r=1}^n \tan^{-1} \left(\frac{2^r - 2^{r-1}}{1+2^r \cdot 2^{r-1}} \right) \\ &= \sum_{r=1}^n [\tan^{-1} (2^r) - \tan^{-1} (2^{r-1})] \\ &= \tan^{-1} (2^n) - \tan^{-1} (1) \\ &= \tan^{-1} (2^n) - \frac{\pi}{4} \end{aligned}$$

$$1. \cos^{-1} \left(\frac{x}{2} \right) + \cos^{-1} \left(\frac{y}{3} \right) = \frac{\pi}{6}$$

$$\Rightarrow \cos^{-1} \left[\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}} \sqrt{1-\frac{y^2}{9}} \right] = \frac{\pi}{6}$$

$$\Rightarrow \left(\frac{xy}{6} - \frac{\sqrt{3}}{2} \right)^2 = \left(1 - \frac{x^2}{4} \right) \left(1 - \frac{y^2}{9} \right)$$

$$\Rightarrow \frac{x^2}{4} - \frac{xy}{2\sqrt{3}} + \frac{y^2}{9} = \frac{1}{4}$$

$$2. \cos^{-1} x + \cos^{-1} \left[\frac{x}{2} + \frac{1}{2} \sqrt{3-3x^2} \right] = \frac{\pi}{3}$$

$$\Rightarrow \cos^{-1} x + \cos^{-1} \left[\frac{1}{2} x + \frac{\sqrt{3}}{2} \sqrt{1-x^2} \right] = \frac{\pi}{3}$$

$$\Rightarrow \cos^{-1} x + \cos^{-1} \frac{1}{2} - \cos^{-1} x = \frac{\pi}{3}$$

Above holds when $\cos^{-1} \frac{1}{2} \geq \cos^{-1} x$.

$$\therefore x \geq \frac{1}{2}$$

$$\therefore x \in \left[\frac{1}{2}, 1 \right]$$

$$3. \sec^{-1} \frac{x}{a} - \sec^{-1} \frac{x}{b} = \sec^{-1} b - \sec^{-1} a$$

$$\Rightarrow \cos^{-1} \frac{a}{x} - \cos^{-1} \frac{b}{x} = \cos^{-1} \frac{1}{b} - \cos^{-1} \frac{1}{a}$$

$$\Rightarrow \cos^{-1} \frac{a}{x} + \cos^{-1} \frac{1}{a} = \cos^{-1} \frac{b}{x} + \cos^{-1} \frac{1}{b}$$

$$\Rightarrow \cos^{-1} \left[\frac{1}{x} - \sqrt{1-\frac{a^2}{x^2}} \sqrt{1-\frac{1}{a^2}} \right] = \cos^{-1} \left[\frac{1}{x} - \sqrt{1-\frac{b^2}{x^2}} \sqrt{1-\frac{1}{b^2}} \right]$$

$$\Rightarrow \frac{1}{x} - \sqrt{1-\frac{1}{a^2} - \frac{a^2}{x^2} + \frac{1}{x^2}} = \frac{1}{x} - \sqrt{1-\frac{b^2}{x^2} - \frac{1}{b^2} + \frac{1}{x^2}}$$

$$\Rightarrow \sqrt{1-\frac{1}{a^2} - \frac{a^2}{x^2} + \frac{1}{x^2}} = \sqrt{1-\frac{b^2}{x^2} - \frac{1}{b^2} + \frac{1}{x^2}}$$

$$\Rightarrow -\frac{1}{a^2} - \frac{a^2}{x^2} = -\frac{1}{b^2} - \frac{b^2}{x^2}$$

$$\Rightarrow \frac{1}{b^2} - \frac{1}{a^2} = \frac{a^2 - b^2}{x^2}$$

$$\Rightarrow x^2 = a^2 b^2$$

$$\therefore x = \pm ab$$

But $x = -ab$ does not satisfy the given equation.
Hence, $x = ab$ is the required solution.

$$4. \sin^{-1} \frac{ax}{c} + \sin^{-1} \frac{bx}{c} = \sin^{-1} x$$

$$\Rightarrow \sin^{-1} \left(\frac{ax}{c} \sqrt{1-\frac{b^2 x^2}{c^2}} + \frac{bx}{c} \sqrt{1-\frac{a^2 x^2}{c^2}} \right) = \sin^{-1} x$$

$$\Rightarrow \frac{ax}{c} \sqrt{1-\frac{b^2 x^2}{c^2}} + \frac{bx}{c} \sqrt{1-\frac{a^2 x^2}{c^2}} = x$$

$$\Rightarrow \frac{a}{c} \sqrt{1 - \frac{b^2 x^2}{c^2}} + \frac{b}{c} \sqrt{1 - \frac{a^2 x^2}{c^2}} = 1 \quad (\text{as } x \neq 0)$$

$$\Rightarrow \frac{a^2}{c^2} \left(1 - \frac{b^2 x^2}{c^2}\right) + \frac{b^2}{c^2} \left(1 - \frac{a^2 x^2}{c^2}\right) + \frac{2ab}{c^2} \sqrt{1 - \frac{a^2 x^2}{c^2}} \sqrt{1 - \frac{b^2 x^2}{c^2}} = 1$$

$$\Rightarrow \frac{a^2 + b^2}{c^2} - \frac{2a^2 b^2 x^2}{c^4} + \frac{2ab}{c^2} \sqrt{1 - \frac{a^2 x^2}{c^2}} \sqrt{1 - \frac{b^2 x^2}{c^2}} = 1$$

$$\Rightarrow \sqrt{1 - \frac{a^2 x^2}{c^2}} \sqrt{1 - \frac{b^2 x^2}{c^2}} = \frac{abx^2}{c^2}$$

$$\Rightarrow \sqrt{c^2 - a^2 x^2} \sqrt{c^2 - b^2 x^2} = abx^2$$

$$\Rightarrow c^4 - c^2(a^2 + b^2)x^2 + a^2 b^2 x^4 = a^2 b^2 x^4$$

$$\Rightarrow c^4 - c^2(a^2 + b^2)x^2 = 0$$

$$\Rightarrow x = \pm 1$$

5. $\cos(\theta - \alpha) = a$ and $\sin(\theta - \beta) = b$

$$\theta = \alpha + \cos^{-1} a = \beta + \sin^{-1} b \quad (\because 0 < \theta - \alpha, \theta - \beta < \pi/2)$$

$$\therefore \alpha - \beta = \sin^{-1} b - \cos^{-1} a$$

$$\Rightarrow \sin(\alpha - \beta) = \sin(\sin^{-1} b - \cos^{-1} a)$$

$$\Rightarrow \sin(\alpha - \beta) = ab - \sqrt{1 - b^2} \sqrt{1 - a^2}$$

$$\Rightarrow \sin(\alpha - \beta) - ab = -\sqrt{1 - b^2} \sqrt{1 - a^2}$$

$$\Rightarrow \sin^2(\alpha - \beta) - 2ab \sin(\alpha - \beta) = 1 - (a^2 + b^2)$$

$$\Rightarrow \cos^2(\alpha - \beta) + 2ab \sin(\alpha - \beta) = a^2 + b^2$$

6. $2 \tan^{-1} 2x = \sin^{-1} \frac{4x}{1 + 4x^2} = \sin^{-1} \frac{2(2x)}{1 + (2x)^2}$

$$\Rightarrow -\frac{\pi}{2} \leq 2 \tan^{-1} 2x \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{4} \leq \tan^{-1} 2x \leq \frac{\pi}{4}$$

$$\Rightarrow -1 \leq 2x \leq 1$$

$$\Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}$$

7. $\tan^{-1} \left(\frac{1 - x^2}{2x} \right) + \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)$

$$= \frac{\pi}{2} - \tan^{-1} \left(\frac{2x}{1 - x^2} \right) + \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)$$

$$= \frac{\pi}{2} - 2 \tan^{-1} x + 2 \tan^{-1} x \quad (\text{as } x \in (0, 1))$$

$$= \frac{\pi}{2}$$

8. $\cos^{-1}(2x^2 - 1) = 2\pi - 2\cos^{-1} x \quad (\text{as } x < 0)$

$$\Rightarrow \cos^{-1}(2x^2 - 1) - 2 \sin^{-1} x = 2\pi - 2 \cos^{-1} x - 2 \sin^{-1} x$$

$$= 2\pi - 2(\cos^{-1} x + \sin^{-1} x)$$

$$= 2\pi - 2 \frac{\pi}{2} = \pi$$

9. $\sin\{2 \sin^{-1}(0.8)\} = \sin\{\sin^{-1}(2 \times 0.8 \sqrt{1 - (0.8)^2})\}$

$$= \sin(\sin^{-1} 0.96) = 0.96$$

Exercises

Single Correct Answer Type

1. (3) $\cos^{-1}(\cos(2 \cot^{-1}(\sqrt{2} - 1))) = \cos^{-1}(\cos(2(67.5^\circ)))$
 $= \cos^{-1}(\cos(135^\circ))$
 $= 135^\circ = \frac{3\pi}{4}$

2. (1) We have

$$\sin^{-1} \left(\cot \left(\sin^{-1} \sqrt{\frac{2 - \sqrt{3}}{4}} + \cos^{-1} \frac{\sqrt{12}}{4} + \sec^{-1} \sqrt{2} \right) \right)$$

$$= \sin^{-1} \left(\cot \left(\sin^{-1} \left(\frac{\sqrt{3} - 1}{2\sqrt{2}} \right) + \cos^{-1} \frac{\sqrt{3}}{2} + \cos^{-1} \frac{1}{\sqrt{2}} \right) \right)$$

$$= \sin^{-1} [\cot(15^\circ + 30^\circ + 45^\circ)]$$

$$= \sin^{-1}(\cot(90^\circ)) = \sin^{-1}(0) = 0$$

3. (3) $\cot^{-1} \frac{n}{\pi} > \frac{\pi}{6}$

or $\frac{n}{\pi} < \cot \frac{\pi}{6}$ [as $\cot^{-1} x$ is a decreasing function]

or $\frac{n}{\pi} < \sqrt{3}$

or $n < \sqrt{3}\pi$

or $n < 5.46$

or maximum value of n is 5

4. (1) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x \quad \forall x \in \mathbb{R} - (-1, 1)$

Also range of $\operatorname{cosec}^{-1}(\operatorname{cosec} x) \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

So combining these two, we get

$$x \in \left[-\frac{\pi}{2}, -1\right] \cup \left[1, \frac{\pi}{2}\right]$$

5. (3) Let $\tan^{-1} 2 = \alpha$ or $\tan \alpha = 2$

and $\cot^{-1} 3 = \beta$ or $\cot \beta = 3$

$$\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$$

$$= \sec^2 \alpha + \operatorname{cosec}^2 \beta$$

$$= 1 + \tan^2 \alpha + 1 + \cot^2 \beta$$

$$= 2 + (2)^2 + (3)^2 = 15$$

6. (4) $f(x) = \tan^{-1} \left(\frac{(\sqrt{12} - 2)x^2}{x^4 + 2x^2 + 3} \right)$

$$= \tan^{-1} \left(\frac{2(\sqrt{3} - 1)}{x^2 + \frac{3}{x^2} + 2} \right)$$

As $x^2 + \frac{3}{x^2} \geq 2\sqrt{3}$

[using A.M. $\geq$ G.M.]

$$\Rightarrow x^2 + \frac{3}{x^2} + 2 \geq 2 + 2\sqrt{3}$$

$$\therefore (f(x))_{\max} = \tan^{-1} \left(\frac{2(\sqrt{3} - 1)}{2(\sqrt{3} + 1)} \right)$$

$$= \frac{\pi}{12}$$

7. (3) We have $\cos^{-1} x + \cos^{-1}(2x) = -\pi$, which is not possible as $\cos^{-1} x$ and $\cos^{-1} 2x$ never take negative values.

Since $\sqrt{x^2 - 3x + 2} \geq 0$

$\Rightarrow 0 \leq \tan^{-1} \sqrt{x^2 - 3x + 2} < \frac{\pi}{2}$

and $\sqrt{4x - x^2 - 3} \geq 0 \Rightarrow 0 < \cos^{-1} \sqrt{4x - x^2 - 3} \leq \frac{\pi}{2}$

Adding, we have $0 < \text{L.H.S.} < \pi$

Therefore, the given equation has no solution.

$\sin^{-1}(x-1)$

$\Rightarrow -1 \leq x-1 \leq 1$

$\Rightarrow 0 \leq x \leq 2$

$\cos^{-1}(x-3)$

$\Rightarrow -1 \leq x-3 \leq 1$

$\Rightarrow 2 \leq x \leq 4$

$\therefore x = 2$

So, $\sin^{-1}(2-1) + \cos^{-1}(2-3) + \tan^{-1} \frac{2}{2-4} = \cos^{-1} k + \pi$

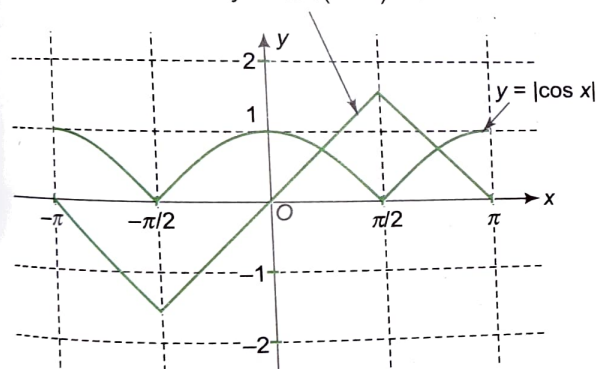
or $\sin^{-1} 1 + \cos^{-1}(-1) + \tan^{-1}(-1) = \cos^{-1} k + \pi$

$\Rightarrow \frac{\pi}{2} + \pi - \frac{\pi}{4} = \cos^{-1} k + \pi$

$\Rightarrow \cos^{-1} k = \frac{\pi}{4}$ or $k = \frac{1}{\sqrt{2}}$

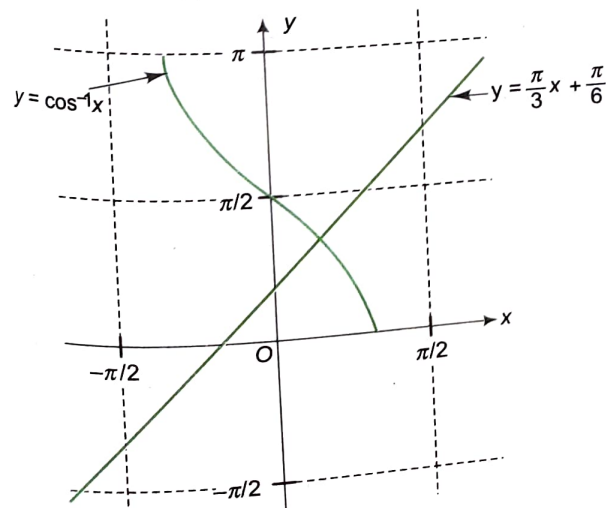
(3) Here $|\cos x| = \sin^{-1}(\sin x)$

$y = \sin^{-1}(\sin x)$



From the graph, the number of solutions is 2.

(2) $3 \cos^{-1} x - \pi x - \frac{\pi}{2} = 0$ or $\cos^{-1} x = \frac{\pi x}{3} + \frac{\pi}{6}$



From the graph, there is only one solution.

12. (3) $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$;
clearly, domain of $f(x)$ is $x = \pm 1$.

Thus, the range is $\{f(1), f(-1)\}$, i.e., $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

13. (4) $\lim_{|x| \rightarrow \infty} \cos(\tan^{-1}(\sin(\tan^{-1} x)))$

$= \cos(\tan^{-1}(\sin(\tan^{-1} \infty)))$

$= \cos\left(\tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)\right)$

$= \cos(\tan^{-1}(1)) = \cos\left(\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$

14. (1) $1 + x^2 \geq 2|x|$

or $\frac{2|x|}{1+x^2} \leq 1$

or $-1 \leq \frac{2x}{1+x^2} \leq 1$

$\Rightarrow \tan^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

15. (4) $[\cot^{-1} x] + 2[\tan^{-1} x] = 0$

$\Rightarrow [\cot^{-1} x] = 0, [\tan^{-1} x] = 0$

or $[\cot^{-1} x] = 2, [\tan^{-1} x] = -1$

Now $[\cot^{-1} x] = 0 \Rightarrow x \in (\cot 1, \infty)$

$[\tan^{-1} x] = 0 \Rightarrow x \in (0, \tan 1)$

Hence, $x \in (\cot 2, \tan 1)$

Therefore,

for $[\cot^{-1} x] = [\tan^{-1} x] = 0, x \in (\cot 1, \tan 1)$

$[\cot^{-1} x] = 2 \Rightarrow x \in (\cot 3, \cot 2]$

$[\tan^{-1} x] = -1 \Rightarrow x \in [-\tan 1, 0)$

Hence, $x \in [-\tan 1, \cot 2]$.

16. (2) Given $\sin^{-1} x + \tan^{-1} x = 2k + 1$

Both $\sin^{-1}$ and $\tan^{-1}$ are increasing functions. So $f(x)$ is increasing function.

The range of the function $\sin^{-1} x + \tan^{-1} x$ is $[f(-1), f(1)]$

or $\left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

Therefore, the integral values of k are -1 and 0 .

17. (2) $\sin \cos^{-1}(\cos(\tan^{-1} x)) = p$

For $x \in \mathbb{R}$, $\tan^{-1} x \in (-\pi/2, \pi/2)$, we have

$\cos(\tan^{-1} x) \in (0, 1]$

$\cos^{-1} \cos(\tan^{-1} x) \in [0, \pi/2)$

$\sin(\cos^{-1}(\cos(\tan^{-1} x))) \in [0, 1)$

18. (2) $0 \leq x^2 + x + 1 \leq 1$ and $0 \leq x^2 + x \leq 1$

$\therefore x = -1, 0$

For $x = -1$

L.H.S. $= 2 \sin^{-1} 1 + \cos^{-1} 0 = \frac{3\pi}{2}$

$\therefore x = -1$ is a solution

For $x = 0$, L.H.S. $= 2 \sin^{-1} 1 + \cos^{-1} 0 = \frac{3\pi}{2}$

Therefore, $x = 0$ is a solution and sum of the solutions $= -1$.

$$19. (1) [\tan(\sin^{-1}x)]^2 > 1$$

$$\Rightarrow \tan(\sin^{-1}x) < -1 \text{ or } \tan(\sin^{-1}x) > 1$$

$$\Rightarrow \frac{\pi}{4} < \sin^{-1}x < \frac{\pi}{2} \text{ or } -\frac{\pi}{2} < \sin^{-1}x < -\frac{\pi}{4}$$

$$\Rightarrow x \in \left(\frac{1}{\sqrt{2}}, 1\right) \text{ or } x \in \left(-1, -\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$$

$$20. (3) \sin^{-1}x = 2 \sin^{-1}a$$

$$\text{Now } -\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} \leq 2 \sin^{-1}a \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{4} \leq \sin^{-1}a \leq \frac{\pi}{4}$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow |a| \leq \frac{1}{\sqrt{2}}$$

$$21. (4) \sin^{-1}x \text{ is defined if } -1 \leq x \leq 1 \text{ and } \sin^{-1}(1-x) \text{ is defined if } -1 \leq 1-x \leq 1 \Rightarrow 0 \leq x \leq 2$$

Therefore, $\sin^{-1}x + n \sin^{-1}(1-x)$ is defined if $0 \leq x \leq 1$ when $0 \leq x \leq 1$, also $0 \leq 1-x \leq 1$. So,

$$0 \leq \sin^{-1}x \leq \frac{\pi}{2} \text{ and } 0 \leq \sin^{-1}(1-x) \leq \frac{\pi}{2}$$

$$\therefore \text{L.H.S.} \geq 0 \text{ and R.H.S.} \leq 0,$$

Which equally holds if L.H.S. = R.H.S. = 0

But L.H.S. = 0 if $\sin^{-1}x$ and $n \sin^{-1}(1-x)$ are simultaneously zero, which is not possible.

$$22. (2) \text{ We have}$$

$$\left| \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \right| < \frac{\pi}{3}$$

$$\Rightarrow -\frac{\pi}{3} < \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) < \frac{\pi}{3}$$

$$\Rightarrow 0 \leq \cos^{-1} \frac{1-x^2}{1+x^2} < \frac{\pi}{3}$$

$$\Rightarrow \frac{1}{2} < \frac{1-x^2}{1+x^2} \leq 1$$

$$\Rightarrow 1+x^2 < 2(1-x^2) \leq 2(1+x^2)$$

$$\Rightarrow 0 \leq x^2 < \frac{1}{3}$$

$$\Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

$$23. (1) \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$$

$$= \sin^{-1}(\sin(12-4\pi)) + \cos^{-1}(\cos(4\pi-12))$$

$$= 12-4\pi+4\pi-12=0$$

$$24. (1) \sin^{-1} \sin \left(\frac{22\pi}{7} \right) = \sin^{-1} \sin \left(3\pi + \frac{\pi}{7} \right) = -\frac{\pi}{7}$$

$$\cos^{-1} \cos \left(\frac{5\pi}{3} \right) = \cos^{-1} \cos \left(2\pi - \frac{\pi}{3} \right) = \frac{\pi}{3}$$

$$\tan^{-1} \tan \left(\frac{5\pi}{7} \right) = \tan^{-1} \tan \left(\pi - \frac{2\pi}{7} \right) = -\frac{2\pi}{7}$$

$$\sin^{-1} \cos(2) = \frac{\pi}{2} - \cos^{-1} \cos 2 = \frac{\pi}{2} - 2$$

$$\therefore \text{Required value} = -\frac{\pi}{7} + \frac{\pi}{3} - \frac{2\pi}{7} + \frac{\pi}{2} - 2$$

$$= \frac{(-18+35)\pi}{42} - 2$$

$$= \frac{17\pi}{42} - 2$$

$$25. (4) \sin^{-1}(\cos(\cos^{-1}(\cos x) + \sin^{-1}(\sin x)))$$

$$= \sin^{-1}(\cos(x + \pi - x))$$

$$= \sin^{-1}(\cos \pi) = \sin^{-1}(-1) = -\frac{\pi}{2}$$

[as $x \in (\pi/2, \pi)$]

$$26. (3) \text{ For } \alpha \in \left(-\frac{3\pi}{2}, -\pi\right), \tan \alpha < 0$$

$$\Rightarrow \tan^{-1}(\cot \alpha) - \cot^{-1}(\tan \alpha)$$

$$= \tan^{-1}(\cot \alpha) - \left[\frac{\pi}{2} - \tan^{-1}(\tan \alpha) \right]$$

$$= \tan^{-1}(\cot \alpha) + \tan^{-1}(\tan \alpha) - \frac{\pi}{2} = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi$$

Also for points in the second quadrant, we have

$$\sin^{-1}(\sin \alpha) + \cos^{-1}(\cos \alpha) = \pi$$

$$27. (1) \tan^{-1} \left[\frac{\cos x}{1 + \sin x} \right] = \tan^{-1} \left[\frac{\sin[(\pi/2) - x]}{1 + \cos[(\pi/2) - x]} \right]$$

$$= \tan^{-1} \left[\frac{2 \sin[(\pi/4) - (x/2)]}{2 \cos^2[(\pi/4) - (x/2)]} \right]$$

$$= \tan^{-1} \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) = \frac{\pi}{4} - \frac{x}{2}$$

$$\Rightarrow -\frac{\pi}{2} < \frac{\pi}{4} - \frac{x}{2} < \frac{\pi}{2}$$

$$\Rightarrow -\frac{3\pi}{4} < -\frac{x}{2} < \frac{\pi}{4}$$

$$\Rightarrow -\frac{\pi}{4} < \frac{x}{2} < \frac{3\pi}{4}$$

$$\Rightarrow -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

$$28. (4) f(x) + f(-x) = 2$$

$$\text{Now } (\sin^{-1}(\sin 8)) = 3\pi - 8 = y$$

$$\text{And } (\tan^{-1}(\tan 8)) = (8 - 3\pi) = -y$$

$$\text{Hence, } f(y) + f(-y) = 2$$

$$\text{Given } f(y) = \alpha, \text{ we have } f(-y) = 2 - \alpha.$$

$$29. (4) \text{ Let } \sin^{-1}x = \theta$$

$$\Rightarrow x = \sin \theta, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$$

$$\text{Now, } \cos^{-1}x = \cos^{-1}(\sin \theta)$$

$$= \cos^{-1} \left(-\cos \left(\frac{3\pi}{2} - \theta \right) \right)$$

$$= \pi - \cos^{-1} \left(\cos \left(\frac{3\pi}{2} - \theta \right) \right)$$

$$= \pi - \left(\frac{3\pi}{2} - \theta \right), \text{ as } 0 \leq \frac{3\pi}{2} - \theta \leq \pi$$

$$= \theta - \frac{\pi}{2} = \sin^{-1} x - \frac{\pi}{2}$$

Hence, $\sin^{-1} x + \cos^{-1} x = 2 \sin^{-1} x - \frac{\pi}{2}$

$$30. (1) \quad 2 \cos^{-1} x = \cot^{-1} \left(\frac{2x^2 - 1}{2x\sqrt{1-x^2}} \right)$$

Put $x = \cos \theta$: L.H.S. $= 2\theta$; $0 \leq \theta \leq \pi$
and $-1 \leq x \leq 1$

$$\text{R.H.S.} = \cot^{-1} \left(\frac{\cos 2\theta}{2 \cos \theta \sin \theta} \right) = \cot^{-1} (\cot 2\theta) = 2\theta \quad \dots(i)$$

if $0 < 2\theta < \pi$

From Eqs. (i) and (ii), we get $0 < \theta < \pi/2$. Thus,
 $x \in (0, 1)$ (ii)

$$31. (2) \quad \tan(\sin^{-1}(\cos(\sin^{-1} x))) \tan(\cos^{-1}(\sin(\cos^{-1} x)))$$

$$= \tan(\sin^{-1}(\cos(\cos^{-1} \sqrt{1-x^2})))$$

$$= \tan\{\cos^{-1}(\sin(\sin^{-1} \sqrt{1-x^2}))\}$$

$$= \tan(\sin^{-1} \sqrt{1-x^2}) \tan(\cos^{-1} \sqrt{1-x^2})$$

$$= \tan(\cos^{-1} x) \tan(\sin^{-1} x)$$

$$= \tan(\cos^{-1} x) \tan(\pi/2 - \cos^{-1} x)$$

$$= \tan(\cos^{-1} x) \cot(\cos^{-1} x) = 1$$

$$32. (3) \quad \text{Let } \tan^{-1}(x) = \theta \text{ or } x = \tan \theta$$

$$\Rightarrow \cos \theta = x \Rightarrow \frac{1}{\sqrt{1+x^2}} = x$$

$$\Rightarrow x^2(1+x^2) = 1 \Rightarrow x^2 = \frac{-1 \pm \sqrt{5}}{2}$$

$$\text{So, } x^2 = \frac{\sqrt{5}-1}{2} \Rightarrow \frac{x^2}{2} = \frac{\sqrt{5}-1}{4}$$

$$\text{Now, } \cos^{-1} \left(\frac{\sqrt{5}-1}{4} \right) = \cos^{-1} \left(\sin \frac{\pi}{10} \right) = \cos^{-1} \left(\cos \frac{2\pi}{5} \right) = \frac{2\pi}{5}$$

$$33. (4) \quad \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = 4^\circ \quad (x \neq 0)$$

$$\Rightarrow \frac{\sqrt{1+x^2}-1}{x} = \tan 4^\circ$$

$$\Rightarrow \sqrt{1+x^2} = 1 + x \tan 4^\circ$$

$$\Rightarrow 1+x^2 = 2x \tan 4^\circ + 1 + x^2 \tan^2 4^\circ$$

$$\Rightarrow x = 0 \text{ or } x = \frac{2 \tan 4^\circ}{1 - \tan^2 4^\circ} = \tan 8^\circ$$

Since $x \neq 0$, we have $x = \tan 8^\circ$.

$$34. (3) \quad \frac{\alpha^3}{2} \operatorname{cosec}^2 \left(\frac{1}{2} \tan^{-1} \frac{\alpha}{\beta} \right) + \frac{\beta^3}{2} \sec^2 \left(\frac{1}{2} \tan^{-1} \frac{\beta}{\alpha} \right)$$

$$= \alpha^3 \frac{1}{1 - \cos \left(\tan^{-1} \left(\frac{\alpha}{\beta} \right) \right)} + \beta^3 \frac{1}{1 + \cos \left(\tan^{-1} \left(\frac{\beta}{\alpha} \right) \right)}$$

$$= \alpha^3 \frac{1}{1 - \cos \left(\cos^{-1} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) \right)} + \beta^3 \frac{1}{1 + \cos \left(\cos^{-1} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \right) \right)}$$

$$= \alpha^3 \frac{1}{1 - \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}} + \beta^3 \frac{1}{1 + \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}}$$

$$= \sqrt{\alpha^2 + \beta^2} \left(\frac{\alpha^3}{\sqrt{\alpha^2 + \beta^2} - \beta} + \frac{\beta^3}{\sqrt{\alpha^2 + \beta^2} + \alpha} \right)$$

$$= \sqrt{\alpha^2 + \beta^2} \left(\alpha^3 \frac{(\sqrt{\alpha^2 + \beta^2} + \beta)}{\alpha^2} + \beta^3 \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)}{\beta^2} \right)$$

$$= \sqrt{\alpha^2 + \beta^2} \left[\alpha (\sqrt{\alpha^2 + \beta^2} + \beta) + \beta (\sqrt{\alpha^2 + \beta^2} - \alpha) \right]$$

$$= \sqrt{\alpha^2 + \beta^2} (\alpha + \beta) \sqrt{\alpha^2 + \beta^2}$$

$$= (\alpha + \beta)(\alpha^2 + \beta^2)$$

$$35. (3) \quad \tan \left(\frac{\pi}{4} + \frac{1}{2} \cos^{-1} x \right) + \tan \left(\frac{\pi}{4} - \frac{1}{2} \cos^{-1} x \right)$$

$$= \frac{1 + \tan \left(\frac{1}{2} \cos^{-1} x \right)}{1 - \tan \left(\frac{1}{2} \cos^{-1} x \right)} + \frac{1 - \tan \left(\frac{1}{2} \cos^{-1} x \right)}{1 + \tan \left(\frac{1}{2} \cos^{-1} x \right)}$$

$$= \frac{\left(1 + \tan \left(\frac{1}{2} \cos^{-1} x \right) \right)^2 + \left(1 - \tan \left(\frac{1}{2} \cos^{-1} x \right) \right)^2}{1 - \tan^2 \left(\frac{1}{2} \cos^{-1} x \right)}$$

$$= 2 \frac{1 + \tan^2 \left(\frac{1}{2} \cos^{-1} x \right)}{1 - \tan^2 \left(\frac{1}{2} \cos^{-1} x \right)}$$

$$= \frac{2}{\cos(\cos^{-1} x)} = \frac{2}{x}$$

$$36. (2) \quad \sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1} x = \frac{\pi}{2} - \sin^{-1} y$$

$$\Rightarrow \sin^{-1} x = \sin^{-1} \sqrt{1-y^2}$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\Rightarrow \frac{1+x^4+y^4}{x^2-x^2y^2+y^2} = \frac{1+(x^2+y^2)^2-2x^2y^2}{1-x^2y^2}$$

$$= \frac{1+1-2x^2y^2}{1-x^2y^2} = 2$$

$$37. (3) \quad 2 \tan^{-1}(\operatorname{cosec} \tan^{-1} x - \tan \cot^{-1} x)$$

$$= 2 \tan^{-1} \left[\operatorname{cosec} \left\{ \operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{x} \right\} - \tan^{-1} \left\{ \tan^{-1} \left(\frac{1}{x} \right) \right\} \right]$$

$$\begin{aligned}
 &= 2 \tan^{-1} \left[\sqrt{\frac{1+x^2}{x}} - \frac{1}{x} \right] = 2 \tan^{-1} \left[\frac{\sqrt{1+x^2}-1}{x} \right] \\
 &= 2 \tan^{-1} \left[\frac{\sec \theta - 1}{\tan \theta} \right] \quad [\text{putting } x = \tan \theta] \\
 &= 2 \tan^{-1} \left[\frac{1 - \cos \theta}{\sin \theta} \right] = 2 \tan^{-1} \left[\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right] \\
 &= 2 \tan^{-1} \tan \frac{\theta}{2} = 2 \times \frac{\theta}{2} = \theta = \tan^{-1} x
 \end{aligned}$$

38. (1) Let $\sin^{-1} a = A$, $\sin^{-1} b = B$, and $\sin^{-1} c = C$

$$\Rightarrow \sin A = a, \sin B = b, \sin C = c$$

$$\text{and } A + B + C = \pi$$

$$\Rightarrow \sin 2A + \sin 2B + \sin 2C$$

$$= 4 \sin A \sin B \sin C$$

$$\Rightarrow \sin A \cos A + \sin B \cos B + \sin C \cos C$$

$$= 2 \sin A \sin B \sin C$$

$$\Rightarrow \sin A \sqrt{1 - \sin^2 A} + \sin B \sqrt{1 - \sin^2 B}$$

$$+ \sin C \sqrt{1 - \sin^2 C} = 2 \sin A \sin B \sin C$$

$$\Rightarrow a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2} = 2abc$$

Alternatively:

$$\text{Let } a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}, c = 1.$$

$$\begin{aligned}
 \text{Then } a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2} \\
 &= \frac{1}{\sqrt{2}} \sqrt{1-\frac{1}{2}} + \frac{1}{\sqrt{2}} \sqrt{1-\frac{1}{2}} + 1\sqrt{1-1} = 1 \\
 &= 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot 1
 \end{aligned}$$

$$39. (4) a \sin^{-1} x - b \cos^{-1} x = c$$

$$\text{We have } b \sin^{-1} x + b \cos^{-1} x = \frac{b\pi}{2}$$

$$\text{Adding } (a+b) \sin^{-1} x = \frac{b\pi}{2} + c$$

$$\Rightarrow \sin^{-1} x = \frac{\left(\frac{b\pi}{2}\right) + c}{a+b} = \frac{b\pi + 2c}{2(a+b)}$$

$$\therefore \cos^{-1} x = \frac{\pi}{2} - \frac{b\pi + 2c}{2(a+b)} = \frac{\pi a - 2c}{2(a+b)}$$

$$\Rightarrow a \sin^{-1} x + b \cos^{-1} x = \frac{\pi ab + c(a-b)}{a+b}$$

$$40. (3) \log_{1/2} \sin^{-1} x > \log_{1/2} \cos^{-1} x$$

$$\Rightarrow \cos^{-1} x > \sin^{-1} x, 0 < x < 1$$

$$\Rightarrow \cos^{-1} x > \frac{\pi}{2} - \cos^{-1} x, 0 < x < 1$$

$$\Rightarrow \cos^{-1} x > \frac{\pi}{4}, 0 < x < 1$$

$$\Rightarrow 0 < x < \frac{1}{\sqrt{2}}$$

$$41. (3) \sin^{-1}(\sin \theta) > \frac{\pi}{2} - \sin^{-1}(\sin \theta)$$

$$\Rightarrow \sin^{-1}(\sin \theta) > \frac{\pi}{4}$$

$$\Rightarrow \sin \theta > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

$$42. (3) |\sin^{-1} x| + |\cos^{-1} x| = \frac{\pi}{2}$$

$$\Rightarrow |\sin^{-1} x| + \cos^{-1} x = \frac{\pi}{2}$$

($\because \cos^{-1} x$ is always non-negative)

$$\Rightarrow |\sin^{-1} x| = \sin^{-1} x$$

$$\Rightarrow \sin^{-1} x \geq 0$$

$$\Rightarrow 0 \leq \sin^{-1} x \leq \frac{\pi}{2}$$

$$\Rightarrow 0 \leq x \leq 1$$

$$43. (1) a\pi^2 = (\sin^{-1} x)^2 - (\cos^{-1} x)^2$$

$$\Rightarrow a\pi^2 = (\sin^{-1} x - \cos^{-1} x)(\sin^{-1} x + \cos^{-1} x)$$

$$\Rightarrow a\pi^2 = (\sin^{-1} x - \cos^{-1} x)(\pi/2)$$

$$\Rightarrow 2a\pi = (\sin^{-1} x - \cos^{-1} x)$$

$$\Rightarrow 2a\pi = (2\sin^{-1} x - \pi/2)$$

$$\text{Now, } -\pi \leq 2\sin^{-1} x \leq \pi$$

$$\Rightarrow -3\pi/2 \leq 2\sin^{-1} x - \pi/2 \leq \pi/2$$

$$\therefore -3\pi/2 \leq 2a\pi \leq \pi/2$$

$$\text{or } -3/4 \leq a \leq 1/4$$

$$44. (2) \sin^{-1} |x-2| + \cos^{-1} (1 - |3-x|) = \frac{\pi}{2}$$

$$\text{or } |x-2| = 1 - |3-x|$$

$$\text{or } |x-2| + |3-x| = |(x-2) + (3-x)|$$

$$\text{or } (x-2)(3-x) \geq 0$$

$$\text{or } 2 \leq x \leq 3$$

$$45. (2) \cos^{-1} \left(\frac{1+x^2}{2x} \right) = \frac{\pi}{2} + (\sin^{-1} x + \cos^{-1} x)$$

$$\Rightarrow \cos^{-1} \left(\frac{1+x^2}{2x} \right) = \pi$$

$$\text{or } \left(\frac{1+x^2}{2x} \right) = \cos \pi = -1$$

$$\text{or } x^2 + 1 + 2x = 0$$

$$\text{or } x = -1$$

$$46. (4) f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x} = \begin{cases} \frac{\pi}{2} & x > 0 \\ -\frac{\pi}{2} & x < 0 \end{cases}$$

$$g(x) = \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \forall x \in [-1, 1]$$

Therefore, common domain is $x \in (0, 1]$.

47. (2) The given equation is

$$ax^2 + \sin^{-1}((x-1)^2 + 1) + \cos^{-1}((x-1)^2 + 1) = 0.$$

$$\text{Now, } -1 \leq (x-1)^2 + 1 \leq 1 \Rightarrow x = 1$$

$$\text{So, we have } a + \frac{\pi}{2} = 0 \quad \text{or} \quad a = -\frac{\pi}{2}$$

$$(3) \text{ Put } \sin^{-1} \frac{5}{x} = A \text{ or } \frac{5}{x} = \sin A$$

$$\sin^{-1} \frac{12}{x} = B \text{ or } \frac{12}{x} = \sin B$$

$$\Rightarrow A + B = \frac{\pi}{2}$$

$$\Rightarrow \sin A = \sin \left(\frac{\pi}{2} - B \right) = \cos B = \sqrt{1 - \sin^2 B}$$

$$\Rightarrow \frac{5}{x} = \sqrt{1 - \frac{144}{x^2}} \text{ or } \frac{169}{x^2} = 1$$

$$\text{or } x^2 = 169 \text{ or } x = 13$$

[$\because x = -13$ does not satisfy the given equation]

$$(4) \cos^{-1} \sqrt{p} + \cos^{-1} \sqrt{1-p} + \cos^{-1} \sqrt{1-q} = \frac{3\pi}{4}$$

$$\text{or } \cos^{-1} \sqrt{p} + \cos^{-1} \sqrt{1-(\sqrt{p})^2} + \cos^{-1} \sqrt{1-q} = \frac{3\pi}{4}$$

$$\text{or } \cos^{-1} \sqrt{p} + \sin^{-1} \sqrt{p} + \cos^{-1} \sqrt{1-q} = \frac{3\pi}{4}$$

$$\text{or } \frac{\pi}{2} + \cos^{-1} \sqrt{1-q} = \frac{3\pi}{4}$$

$$\text{or } \cos^{-1} \sqrt{1-q} = \frac{3\pi}{4} - \frac{\pi}{2}$$

$$\text{or } \cos^{-1} \sqrt{1-q} = \frac{\pi}{4}$$

$$\text{or } \sqrt{1-q} = \cos \frac{\pi}{4}$$

$$\text{or } \sqrt{1-q} = \frac{1}{\sqrt{2}}$$

$$\text{or } q = \frac{1}{2}$$

$$50. (3) \text{ From the given equation } \sin^2 \theta - 2 \sin \theta + 3 = 5^{\sec^2 \theta} + 1, \text{ we get}$$

$$(\sin \theta - 1)^2 + 2 = 5^{\sec^2 \theta} + 1$$

$$\text{L.H.S.} \leq 6, \text{ R.H.S.} \geq 6$$

Possible solution is $\sin \theta = -1$ when L.H.S. = R.H.S.

$$\Rightarrow \cos^2 \theta = 0$$

$$\Rightarrow \cos^2 \theta - \sin \theta = 1$$

$$51. (1) \frac{\pi}{2} - \cos^{-1} \cos \left(\frac{2(x^2 + 5|x| + 3) - 2}{x^2 + 5|x| + 3} \right)$$

$$= \cot \cot^{-1} \left(\frac{2}{9|x|} - 2 \right) + \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} - 2 + \frac{2}{x^2 + 5|x| + 3} = \frac{2}{9|x|} - 2 + \frac{\pi}{2}$$

$$\Rightarrow |x|^2 - 4|x| + 3 = 0$$

$$\Rightarrow |x| = 1, 3 \text{ or } x = \pm 1, \pm 3$$

$$52. (3) a - \cot^{-1} 3x = 2 \tan^{-1} 3x + \cos^{-1} x \sqrt{3} + \sin^{-1} x \sqrt{3}$$

Clearly, given equation is meaningful when $-1 \leq x \sqrt{3} \leq 1$

Given equation becomes

$$a = \tan^{-1} 3x + (\tan^{-1} 3x + \cot^{-1} 3x) + (\cos^{-1} x \sqrt{3} + \sin^{-1} x \sqrt{3})$$

$$= \tan^{-1} 3x + \pi/2 + \pi/2$$

$$= \pi + \tan^{-1} 3x$$

$$\text{Now } -1 \leq x \sqrt{3} \leq 1$$

$$\Rightarrow -\sqrt{3} \leq 3x \leq \sqrt{3}$$

$$\Rightarrow -\frac{\pi}{3} \leq \tan^{-1} 3x \leq \frac{\pi}{3}$$

$$\Rightarrow \frac{2\pi}{3} \leq \pi + \tan^{-1} 3x \leq \frac{4\pi}{3}$$

$$\Rightarrow a \in \left[\frac{2\pi}{3}, \frac{4\pi}{3} \right]$$

$$53. (1) \text{ Let } \sqrt{\tan \alpha} = \tan x. \text{ Then}$$

$$u = \cot^{-1} (\tan x) - \tan^{-1} (\tan x)$$

$$= \frac{\pi}{2} - x - x = \frac{\pi}{2} - 2x$$

$$\text{or } 2x = \frac{\pi}{2} - u$$

$$\text{or } x = \frac{\pi}{4} - \frac{u}{2}$$

$$\text{or } \tan x = \tan \left(\frac{\pi}{4} - \frac{u}{2} \right)$$

$$\Rightarrow \sqrt{\tan \alpha} = \tan \left(\frac{\pi}{4} - \frac{u}{2} \right)$$

$$54. (3) \sin^{-1} \sqrt{1-x^2} + \cos^{-1} x = \cot^{-1} \frac{\sqrt{1-x^2}}{x} - \sin^{-1} x$$

$$\text{or } \frac{\pi}{2} + \sin^{-1} \sqrt{1-x^2} = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$$

$$\tan^{-1} \frac{\sqrt{1-x^2}}{x} = -\sin^{-1} \sqrt{1-x^2}$$

$$\Rightarrow x \in [-1, 0) \cup \{1\}$$

$$55. (4) \cos^{-1} \left(\sqrt{\frac{2}{3}} \right) - \cos^{-1} \left(\frac{\sqrt{6}+1}{2\sqrt{3}} \right)$$

$$= \tan^{-1} \left(\frac{1}{\sqrt{2}} \right) - \tan^{-1} \left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}} \right)$$

$$= \tan^{-1} \left(\frac{1}{\sqrt{2}} \right) - \left[\tan^{-1} \sqrt{3} - \tan^{-1} \sqrt{2} \right]$$

$$= \left(\tan^{-1} \frac{1}{\sqrt{2}} + \tan^{-1} \sqrt{2} \right) - \tan^{-1} \sqrt{3}$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

$$56. (1) \theta = \tan^{-1} (2 \tan^2 \theta) - \tan^{-1} \left(\frac{1}{3} \tan \theta \right) = \tan^{-1} \frac{2 \tan^2 \theta - \frac{1}{3} \tan \theta}{1 + \frac{2}{3} \tan^2 \theta}$$

$$\Rightarrow \tan \theta = \frac{6 \tan^2 \theta - \tan \theta}{3 + 2 \tan^2 \theta}$$

$$\Rightarrow 1 = \frac{6 \tan \theta - 1}{3 + 2 \tan^2 \theta} \text{ or } \tan \theta = 0$$

$$\Rightarrow 2 \tan^4 \theta - 6 \tan^2 \theta + 4 = 0 \text{ or } \tan \theta = 0$$

$$\Rightarrow (\tan^2 \theta - 1)^2 (\tan^2 \theta + 2) = 0 \text{ or } \tan \theta = 0$$

$$\Rightarrow \tan \theta = 1, \tan \theta = -2, \tan \theta = 0$$

$$57. (2) \quad y = \tan^{-1} \frac{1}{2} + \tan^{-1} b, (0 < b < 1) \text{ and } 0 < y \leq \frac{\pi}{4}$$

$$\tan^{-1} b = y - \tan^{-1} \frac{1}{2}$$

Thus, maximum value of b occurs when y is maximum.

$$\therefore \tan^{-1} b = \frac{\pi}{4} - \tan^{-1} \frac{1}{2}$$

$$\Rightarrow \tan^{-1} b = \tan^{-1} 1 - \tan^{-1} \frac{1}{2}$$

$$\Rightarrow \tan^{-1} b = \tan^{-1} \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}}$$

$$\Rightarrow \tan^{-1} b = \tan^{-1} \frac{1}{3}$$

$$\Rightarrow b_{\max} = \frac{1}{3}$$

58. (4) $x = 1, y = 1$ is not a solution of the given equation.

Suppose $(x, y) \neq (1, 1)$

$$\text{Then } \frac{\frac{1}{x} + \frac{1}{y}}{1 - \frac{1}{xy}} = \frac{1}{z}$$

$$\Rightarrow x(z - y) = -(1 + yz)$$

$$\Rightarrow x = -\frac{(1 + yz)}{z - y}$$

If $y = n + 1, z = n$ then $x = n^2 + n + 1$.

All such numbers are solutions of the given equation

59. (4) According to the question,

$$\sin^{-1} \frac{2}{\sqrt{5}} + \sin^{-1} \frac{3}{\sqrt{10}} + \sin^{-1} \alpha = \pi$$

$$\Rightarrow \tan^{-1} 2 + \tan^{-1} 3 + \tan^{-1} \left(\frac{\alpha}{\sqrt{1 - \alpha^2}} \right) = \pi$$

$$\Rightarrow \pi + \tan^{-1} \frac{2 + 3}{1 - (2)(3)} + \tan^{-1} \left(\frac{\alpha}{\sqrt{1 - \alpha^2}} \right) = \pi$$

$$\Rightarrow \tan^{-1} \left(\frac{\alpha}{\sqrt{1 - \alpha^2}} \right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{\alpha}{\sqrt{1 - \alpha^2}} = 1$$

$$\Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

$$60. (3) \quad \tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \frac{\pi}{2}$$

$$\text{or } \tan^{-1}(1 + x) = \frac{\pi}{2} - \tan^{-1}(1 - x)$$

$$= \cot^{-1}(1 - x)$$

$$= \tan^{-1} \left(\frac{1}{1 - x} \right)$$

$$\text{or } 1 + x = \frac{1}{1 - x}$$

$$\text{or } 1 - x^2 = 1$$

$$\text{or } x = 0$$

$$61. (2) \quad \tan^{-1} \frac{1}{1 + 2x} + \tan^{-1} \frac{1}{1 + 4x} = \tan^{-1} \frac{2}{x^2}$$

$$\text{or } \tan^{-1} \left[\frac{\frac{1}{1 + 2x} + \frac{1}{1 + 4x}}{1 - \frac{1}{1 + 2x} \cdot \frac{1}{1 + 4x}} \right] = \tan^{-1} \frac{2}{x^2}$$

$$\text{or } \frac{2 + 6x}{6x + 8x^2} = \frac{2}{x^2}$$

$$\text{or } 6x^3 - 14x^2 - 12x = 0$$

$$\text{or } x(x - 3)(3x + 2) = 0$$

$$\text{or } x = 3 \text{ or } x = -2/3$$

But for $x = -2/3$, L.H.S. < 0 and R.H.S. > 0

Hence, the only solution is $x = 3$.

(as $x \neq 0$)

$$62. (2) \quad \cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$$

$$\text{or } \frac{\pi}{2} - \tan^{-1} x + \frac{\pi}{2} - \tan^{-1} y + \frac{\pi}{2} - \tan^{-1} z = \frac{\pi}{2}$$

$$\text{or } \tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$$

$$\text{or } \tan^{-1} x + \tan^{-1} y = \pi - \tan^{-1} z$$

$$\text{or } \tan(\tan^{-1} x + \tan^{-1} y) = \tan(\pi - \tan^{-1} z)$$

$$\text{or } \frac{x + y}{1 - xy} = -z$$

$$\text{or } x + y + z = xyz$$

63. (2) We have

$$\frac{xy}{zr} \cdot \frac{yz}{xr} = \frac{y^2}{r^2} = \frac{y^2}{x^2 + y^2 + z^2} < 1$$

$$\Rightarrow \tan^{-1} \left(\frac{xy}{zr} \right) + \tan^{-1} \left(\frac{yz}{xr} \right) + \tan^{-1} \left(\frac{xz}{yr} \right)$$

$$= \tan^{-1} \left(\frac{\frac{xy}{zr} + \frac{yz}{xr}}{1 - \frac{xy}{zr} \cdot \frac{yz}{xr}} \right) + \tan^{-1} \left(\frac{xz}{yr} \right)$$

$$= \tan^{-1} \left(\frac{\frac{y(x^2 + z^2)}{r^2 - y^2}}{\frac{xz}{r^2}} \right) + \tan^{-1} \left(\frac{xz}{yr} \right)$$

$$= \tan^{-1} \left(\frac{yr(x^2 + z^2)}{xz} \right) + \tan^{-1} \left(\frac{xz}{yr} \right)$$

$$= \tan^{-1} \left(\frac{yr}{xz} \right) + \tan^{-1} \left(\frac{xz}{yr} \right) = \frac{\pi}{2}$$

$$64. (2) \quad \tan^{-1} \left(\frac{x \cos \theta}{1 - x \sin \theta} \right) - \cot^{-1} \left(\frac{\cos \theta}{x - \sin \theta} \right)$$

$$= \tan^{-1} \left(\frac{x \cos \theta}{1 - x \sin \theta} \right) - \tan^{-1} \left(\frac{x - \sin \theta}{\cos \theta} \right)$$

$$= \tan^{-1} \left(\frac{\frac{x \cos \theta}{1 - x \sin \theta} - \frac{x - \sin \theta}{\cos \theta}}{1 + \left(\frac{x \cos \theta}{1 - x \sin \theta} \right) \left(\frac{x - \sin \theta}{\cos \theta} \right)} \right)$$

$$\begin{aligned}
&= \tan^{-1} \left(\frac{x \cos^2 \theta - x + \sin \theta + x^2 \sin \theta - x \sin^2 \theta}{\cos \theta - x \cos \theta \sin \theta + x^2 \cos \theta - x \cos \theta \sin \theta} \right) \\
&= \tan^{-1} \left(\frac{-x \sin^2 \theta + \sin \theta + x^2 \sin \theta - x \sin^2 \theta}{\cos \theta - 2x \cos \theta \sin \theta + x^2 \cos \theta} \right) \\
&= \tan^{-1} \left(\frac{-2x \sin^2 \theta + \sin \theta + x^2 \sin \theta}{\cos \theta - 2x \cos \theta \sin \theta + x^2 \cos \theta} \right) \\
&= \tan^{-1} \left(\frac{\sin \theta (-2x \sin \theta + 1 + x^2)}{\cos \theta (1 - 2x \sin \theta + x^2)} \right) \\
&= \tan^{-1} (\tan \theta) \\
&= \theta
\end{aligned}$$

$$(1) \cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$$

$$\text{or } \tan^{-1} \left(\frac{1}{\sqrt{\cos \alpha}} \right) - \tan^{-1}(\sqrt{\cos \alpha}) = x$$

$$\text{or } \tan^{-1} \frac{\frac{1}{\sqrt{\cos \alpha}} - \sqrt{\cos \alpha}}{1 + \frac{1}{\sqrt{\cos \alpha}} \sqrt{\cos \alpha}} = x$$

$$\text{or } \tan^{-1} \frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}} = x$$

$$\text{or } \tan x = \frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}}$$

$$\text{or } \cot x = \frac{2\sqrt{\cos \alpha}}{1 - \cos \alpha}$$

$$\text{or } \operatorname{cosec} x = \sqrt{1 + \frac{4 \cos \alpha}{(1 - \cos \alpha)^2}} = \frac{1 + \cos \alpha}{1 - \cos \alpha}$$

$$\text{or } \sin x = \frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{2 \sin^2(\alpha/2)}{2 \cos^2(\alpha/2)} = \tan^2 \frac{\alpha}{2}$$

$$(3) \sin^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}} \right) = \tan^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{1 + \sqrt{r(r-1)}} \right)$$

$$\begin{aligned}
\therefore \sum_{r=1}^n \sin^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}} \right) \\
= \sum_{r=1}^n (\tan^{-1} \sqrt{r} - \tan^{-1} \sqrt{r-1}) = \tan^{-1} \sqrt{n}
\end{aligned}$$

$$(1) \text{ We have } \sum_{m=1}^n \tan^{-1} \left(\frac{2m}{m^4 + m^2 + 2} \right)$$

$$= \sum_{m=1}^n \tan^{-1} \left(\frac{2m}{1 + (m^2 + m + 1)(m^2 - m + 1)} \right)$$

$$= \sum_{m=1}^n \tan^{-1} \left(\frac{(m^2 + m + 1) - (m^2 - m + 1)}{1 + (m^2 + m + 1)(m^2 - m + 1)} \right)$$

$$= \sum_{m=1}^n [\tan^{-1}(m^2 + m + 1) - \tan^{-1}(m^2 - m + 1)]$$

$$\begin{aligned}
&= (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 7 - \tan^{-1} 3) + \\
&\quad (\tan^{-1} 13 - \tan^{-1} 7) + \dots + [\tan^{-1}(n^2 + n + 1) - \tan^{-1}(n^2 - n + 1)] \\
&= \tan^{-1}(n^2 + n + 1) - \tan^{-1} 1 \\
&= \tan^{-1} \left(\frac{n^2 + n}{2 + n^2 + n} \right)
\end{aligned}$$

68. (2) Let us find the value of general term of the series

$$S = 7 + 19 + 39 + 67 + \dots + T_n$$

$$S = 7 + 19 + 39 + \dots + T_{n-1} + T_n$$

$$0 = 7 + 12 + 20 + 28 + \dots (n \text{ terms}) - T_n \quad (\text{subtracting})$$

$$\therefore T_n = 7 + \frac{(n-1)}{2} [24 + 8(n-2)] = 4n^2 + 3$$

$$\therefore T_n = \tan^{-1} \frac{4}{4n^2 + 3} = \tan^{-1} \frac{1}{n^2 + \frac{3}{4}}$$

$$= \tan^{-1} \frac{1}{1 + \left(n^2 - \frac{1}{4}\right)}$$

$$= \tan^{-1} \left[\frac{\left(n + \frac{1}{2}\right) - \left(n - \frac{1}{2}\right)}{1 + \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)} \right]$$

$$= \tan^{-1} \left(n + \frac{1}{2} \right) - \tan^{-1} \left(n - \frac{1}{2} \right)$$

Hence,

$$\therefore S_n = \sum_{r=1}^n T_r = \tan^{-1} \left(n + \frac{1}{2} \right) - \tan^{-1} \frac{1}{2}$$

$$\therefore S_n = \tan^{-1} \infty - \tan^{-1} \frac{1}{2}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1} 1 + \cot^{-1} 3$$

$$69. (2) T_n = \sec^{-1} \sqrt{\frac{(n^2 + 1)(n^2 - 2n + 2)}{(n^2 - n + 1)^2}}$$

$$\Rightarrow \sec^2 T_n = \frac{(n^2 + 1)(n^2 - 2n + 2)}{(n^2 - n + 1)^2}$$

$$\Rightarrow \sec^2 T_n = \frac{(n^2 + 1)^2 + (n^2 + 1) - 2n(n^2 + 1)}{(n^2 - n + 1)^2}$$

$$\Rightarrow \sec^2 T_n = \frac{1 + (n^2 + 1 - n)^2}{(n^2 - n + 1)^2}$$

$$\Rightarrow \tan T_n = \frac{1}{n^2 - n + 1}$$

$$\Rightarrow \tan T_n = \frac{n - (n-1)}{1 + n(n-1)} = \tan^{-1} n - \tan^{-1}(n-1)$$

$$\therefore S = T_1 + T_2 + T_3 + \dots + T_n$$

$$\therefore S = \tan^{-1} n$$

$$70. (3) \frac{1}{2} \sin^{-1} \frac{3 \sin 2\theta}{5 + 4 \cos 2\theta} = \tan^{-1} x$$

$$\Rightarrow \sin^{-1} \frac{\frac{6 \tan \theta}{1 + \tan^2 \theta}}{5 + 4 \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)} = \sin^{-1} \frac{2x}{1 + x^2}$$

$$\Rightarrow \sin^{-1} \frac{6 \tan \theta}{9 + \tan^2 \theta} = \sin^{-1} \frac{2x}{1+x^2}$$

$$\Rightarrow \sin^{-1} \frac{2 \left(\frac{\tan \theta}{3} \right)}{1 + \left(\frac{\tan \theta}{3} \right)^2} = \sin^{-1} \frac{2x}{1+x^2}$$

$$\Rightarrow x = \frac{\tan \theta}{3}$$

$$71. (1) \quad 2 \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan \frac{\theta}{2} \right]$$

$$= \cos^{-1} \left[\frac{1 - \left(\frac{a-b}{a+b} \right) \tan^2 \frac{\theta}{2}}{1 + \left(\frac{a-b}{a+b} \right) \tan^2 \frac{\theta}{2}} \right]$$

$$\left[\because 2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2} \right]$$

$$= \cos^{-1} \left[\frac{(a+b) - (a-b) \tan^2 \frac{\theta}{2}}{(a+b) + (a-b) \tan^2 \frac{\theta}{2}} \right]$$

$$= \cos^{-1} \left[\frac{a \left(1 - \tan^2 \frac{\theta}{2} \right) + b \left(1 + \tan^2 \frac{\theta}{2} \right)}{a \left(1 + \tan^2 \frac{\theta}{2} \right) + b \left(1 - \tan^2 \frac{\theta}{2} \right)} \right]$$

$$= \cos^{-1} \left[\frac{\frac{a \left(1 - \tan^2 \frac{\theta}{2} \right)}{1 + \tan^2 \frac{\theta}{2}} + b}{a + b \left(\frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \right)} \right]$$

$$= \cos^{-1} \left[\frac{a \cos \theta + b}{a + b \cos \theta} \right]$$

$$72. (4) \quad \text{Since } \sin^{-1} \left(\frac{2x}{1+x^2} \right) = 2 \tan^{-1} x \text{ for } x \in (-1, 1)$$

$$\sin^{-1} \left(\frac{2a}{1+a^2} \right) + \sin^{-1} \left(\frac{2b}{1+b^2} \right) = 2 \tan^{-1} x$$

$$\Rightarrow 2 \tan^{-1} a + 2 \tan^{-1} b = 2 \tan^{-1} x$$

$$\text{or } \tan^{-1} a + \tan^{-1} b = \tan^{-1} x$$

$$\text{or } \tan^{-1} \left(\frac{a+b}{1-ab} \right) = \tan^{-1} x$$

$$\text{or } x = \frac{a+b}{1-ab}$$

73. (1) The given equation is

$$3(2 \tan^{-1} x) - 4(2 \tan^{-1} x) + 2(2 \tan^{-1} x) = \pi/3$$

$$\therefore 2 \tan^{-1} x = \pi/3$$

$$\therefore \tan^{-1} x = \pi/6$$

$$\therefore x = \frac{1}{\sqrt{3}}$$

$$74. (3) \quad x_1 = 2 \tan^{-1} \left(\frac{1+x}{1-x} \right)$$

$$\text{and } x_2 = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

$$= \tan^{-1} \left(\frac{1-x^2}{2x} \right)$$

$$\text{Now } \frac{1+x}{1-x} > 1$$

$$\Rightarrow x_1 = \pi + \tan^{-1} \left(\frac{2 \left(\frac{1+x}{1-x} \right)}{1 - \left(\frac{1+x}{1-x} \right)^2} \right)$$

$$= \pi + \tan^{-1} \left(\frac{1-x^2}{-2x} \right)$$

$$= \pi - \tan^{-1} \left(\frac{1-x^2}{2x} \right)$$

$$\Rightarrow x_1 + x_2 = \pi$$

$$75. (1) \quad \text{Let } f(x) = x^3 + bx^2 + cx + 1.$$

$$f(0) = 1 > 0, f(-1) = b - c < 0$$

$$\text{So, } \alpha \in (-1, 0)$$

$$\text{So, } 2 \tan^{-1} (\operatorname{cosec} \alpha) + \tan^{-1} (2 \sin \alpha \sec^2 \alpha)$$

$$= 2 \tan^{-1} \left(\frac{1}{\sin \alpha} \right) + \tan^{-1} \left(\frac{2 \sin \alpha}{1 - \sin^2 \alpha} \right)$$

$$= 2 \left[\tan^{-1} \left(\frac{1}{\sin \alpha} \right) + \tan^{-1} (\sin \alpha) \right]$$

$$= 2 \left(-\frac{\pi}{2} \right) = -\pi$$

(as $\sin \alpha < 0$)

$$76. (2) \quad \text{Let } x = \sin \theta \text{ and } \sqrt{x} = \sin \phi, \text{ where } x \in [0, 1]$$

$$\Rightarrow \theta, \phi \in [0, \pi/2]$$

$$\Rightarrow \theta - \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\text{Now, } \sin^{-1} \left(x \sqrt{1-x} - \sqrt{x} \sqrt{1-x^2} \right)$$

$$= \sin^{-1} \left(\sin \theta \sqrt{1 - \sin^2 \phi} - \sin \phi \sqrt{1 - \sin^2 \theta} \right)$$

$$= \sin^{-1} (\sin \theta \cos \phi - \sin \phi \cos \theta)$$

$$= \sin^{-1} \sin(\theta - \phi) = \theta - \phi$$

$$= \sin^{-1}(x) - \sin^{-1}(\sqrt{x})$$

$$77. (4) \quad \text{We have } \cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$$

$$\text{or } x = \cos \left(\cos^{-1} \frac{y}{2} + \alpha \right)$$

$$= \cos \left(\cos^{-1} \frac{y}{2} \right) \cos \alpha - \sin \left(\cos^{-1} \frac{y}{2} \right) \sin \alpha$$

$$= \frac{y}{2} \cos \alpha - \sqrt{1 - \frac{y^2}{4}} \sin \alpha$$

$$\text{or } 2x = y \cos \alpha - \sin \alpha \sqrt{4 - y^2}$$

$$\text{or } 2x - y \cos \alpha = -\sin \alpha \sqrt{4 - y^2}$$

Squaring, we get

$$4x^2 + y^2 \cos^2 \alpha - 4xy \cos \alpha = 4 \sin^2 \alpha - y^2 \sin^2 \alpha$$

$$\text{or } 4x^2 - 4xy \cos \alpha + y^2 = 4 \sin^2 \alpha$$

(2) Since $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

$$\therefore \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$$

$$\Rightarrow \sin^{-1} \left(x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) = \pi - \sin^{-1}(z)$$

$$\Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin(\pi - \sin^{-1}(z))$$

$$= \sin(\sin^{-1} z) = z$$

$$\text{or } x^2(1-y^2) = z^2 + y^2(1-x^2) - 2zy\sqrt{1-x^2}$$

$$\text{or } (x^2 - y^2 - z^2)^2 = 4y^2 z^2 (1-x^2)$$

$$\text{or } x^4 + y^4 + z^4 - 2x^2 y^2 - 2x^2 z^2 + 2y^2 z^2 = 4y^2 z^2 - 4x^2 y^2 z^2$$

$$\text{or } x^4 + y^4 + z^4 + 4x^2 y^2 z^2 = 2(x^2 y^2 + y^2 z^2 + z^2 x^2)$$

$$\Rightarrow K = 2$$

9. (2) Let $x = \sin \theta$, where $-\frac{1}{2} \leq x \leq 1$

$$\text{or } -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}. \text{ Then,}$$

$$f(x) = \sin^{-1} \left(\frac{\sqrt{3}}{2} x - \frac{1}{2} \sqrt{1-x^2} \right)$$

$$= \sin^{-1} \left(\frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta \right)$$

$$= \sin^{-1} \left(\sin \left(\theta - \frac{\pi}{6} \right) \right)$$

$$= \theta - \frac{\pi}{6} = \sin^{-1} x - \frac{\pi}{6} \quad \left[\because \theta - \frac{\pi}{6} \in \left(-\frac{\pi}{3}, \frac{\pi}{3} \right) \right]$$

$$80. (4) \quad 2^{2\pi/\sin^{-1} x} - 2(a+2)2^{\pi/\sin^{-1} x} + 8a < 0$$

$$(2^{\pi/\sin^{-1} x} - 4)(2^{\pi/\sin^{-1} x} - 2a) < 0$$

$$\text{Now } 2^{\pi/\sin^{-1} x} \in \left(0, \frac{1}{4} \right] \cup [4, \infty)$$

$$\text{for } 2^{\pi/\sin^{-1} x} \in \left(0, \frac{1}{4} \right]. \text{ We have}$$

$$(2^{\pi/\sin^{-1} x} - 4) < 0$$

$$\therefore 2^{\pi/\sin^{-1} x} - 2a > 0$$

$$\text{or } 2a < 2^{\pi/\sin^{-1} x}$$

$$\text{or } 2a < \frac{1}{4}$$

$$\text{or } 0 \leq a < \frac{1}{8}$$

Similarly, for $2^{\pi/\sin^{-1} x} \in [4, \infty)$, $a > 2$, we get

$$\text{So, } a \in \left[0, \frac{1}{8} \right) \cup (2, \infty)$$

$$\text{or } x = -3/2, -1/3$$

For $x = -3/2$, $\cos^{-1} x$ is not defined as domain of $\cos^{-1} x$ is $[-1, 1]$ and for $x = -1/3$, $\operatorname{cosec}^{-1} x$ is not defined as domain of $\operatorname{cosec}^{-1} x$ is $R - (-1, 1)$. However, $\cot^{-1} x$ is defined for both of these values as domain of $\cot^{-1} x$ is R .

2. (1), (2), (3).

$$\text{Let } \tan^{-1}(-2) = \theta \quad \text{or } \tan \theta = -2$$

$$\Rightarrow \theta \in (-\pi/2, 0) \quad \text{or } 2\theta \in (-\pi, 0)$$

$$\cos(-2\theta) = \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{-3}{5}$$

$$\text{or } -2\theta = \cos^{-1} \left(\frac{-3}{5} \right) = \pi - \cos^{-1} \frac{3}{5}$$

$$\text{or } 2\theta = -\pi + \cos^{-1} \frac{3}{5} = -\pi + \tan^{-1} \frac{4}{3}$$

$$= -\pi + \cot^{-1} \frac{3}{4} = -\pi + \frac{\pi}{2} - \tan^{-1} \frac{3}{4}$$

$$= -\frac{\pi}{2} - \tan^{-1} \frac{3}{4} = -\frac{\pi}{2} + \tan^{-1} \left(-\frac{3}{4} \right)$$

3. (2), (3), (4).

$$\cos \left(-\frac{14\pi}{5} \right) = \cos \frac{14\pi}{5} = \cos \frac{4\pi}{5}$$

$$\text{Hence, } \cos \frac{1}{2} \cos^{-1} \left(\cos \frac{4\pi}{5} \right) = \cos \frac{2\pi}{5}$$

4. (1), (2), (3).

$$(1) \sin \left(\tan^{-1} 3 + \tan^{-1} \frac{1}{3} \right) = \sin \frac{\pi}{2} = 1$$

$$(2) \cos \left(\frac{\pi}{2} - \sin^{-1} \frac{3}{4} \right) = \cos \left(\cos^{-1} \frac{3}{4} \right) = \frac{3}{4}$$

$$(3) \sin \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right)$$

$$\text{Let } \sin^{-1} \frac{\sqrt{63}}{8} = \theta. \text{ Then,}$$

$$\sin \theta = \frac{\sqrt{63}}{8} \Rightarrow \cos \theta = \frac{1}{8}$$

$$\text{We have } \cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \frac{3}{4}$$

$$\Rightarrow \sin \frac{\theta}{4} = \sqrt{\frac{1 - \cos \frac{\theta}{2}}{2}} = \frac{1}{2\sqrt{2}}$$

$$\text{Now, } \log_2 \sin \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right) = \log_2 \frac{1}{2\sqrt{2}} = -\frac{3}{2}$$

$$(4) \cos^{-1} \frac{\sqrt{5}}{3} = \theta \quad \text{or } \cos \theta = \frac{\sqrt{5}}{3}$$

$$\therefore \tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{3 - \sqrt{5}}{2}, \text{ which is irrational.}$$

5. (1), (2), (3)

$$(1) \cos(\tan^{-1}(\tan(4 - \pi)))$$

$$= \cos(4 - \pi) = \cos(\pi - 4) = -\cos 4 > 0$$

$$(2) \sin(\cot^{-1}(\cot(4 - \pi)))$$

$$= \sin(4 - \pi) = -\sin 4 > 0 \text{ (as } \sin 4 < 0)$$

Multiple Correct Answers Type

1. (2), (3), (4).

$$6x^2 + 11x + 3 = 0$$

$$\text{or } (2x + 3)(3x + 1) = 0$$

$$(3) \quad \tan(\cos^{-1}(\cos(2\pi - 5)))$$

$$= \tan(2\pi - 5) = -\tan 5 > 0 \text{ (as } \tan 5 < 0)$$

$$(4) \quad \cot(\sin^{-1}(\sin(\pi - 4))) = \cot(\pi - 4) = -\cot 4 < 0$$

6. (1), (2), (3)

$$\tan^{-1}x, x < 0$$

$$\text{Let } x = -y, y > 0$$

$$\therefore \tan^{-1}x = -\tan^{-1}y$$

$$= -\cos^{-1} \frac{1}{\sqrt{1+y^2}} \quad \dots(i)$$

$$= -\sin^{-1} \frac{y}{\sqrt{1+y^2}} \quad \dots(ii)$$

$$= -\operatorname{cosec}^{-1} \frac{\sqrt{1+y^2}}{y} \quad \dots(iii)$$

$$= -\cot^{-1} \frac{1}{y} \quad \dots(iv)$$

$$\text{From (i), } \tan^{-1}x = -\cos^{-1} \frac{1}{\sqrt{1+x^2}}$$

$$\text{From (ii), } \tan^{-1}x = -\sin^{-1} \frac{-x}{\sqrt{1+x^2}} = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$$

$$\text{From (iii), } \tan^{-1}x = -\operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{-x} = \operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{x}$$

$$\begin{aligned} \text{From (iv), } \tan^{-1}x &= -\cot^{-1} \frac{1}{y} \\ &= -\cot^{-1} \frac{1}{-x} = -\left(\pi - \cot^{-1} \frac{1}{x}\right) = -\pi + \cot^{-1} \frac{1}{x} \end{aligned}$$

7. (1), (2), (3), (4)

$$\cos^{-1}x = \sec^{-1} \frac{1}{x} \text{ for all } x \in [-1, 0)$$

$$\text{Let } x = -y$$

$$\therefore \cos^{-1}x = \cos^{-1}(-y) = \pi - \cos^{-1}y$$

$$= \pi - \sin^{-1} \sqrt{1-y^2} \quad \dots(i)$$

$$= \pi - \tan^{-1} \frac{\sqrt{1-y^2}}{y} \quad \dots(ii)$$

$$\text{From (i), } \cos^{-1}x = \pi - \sin^{-1} \sqrt{1-y^2} = \pi - \sin^{-1} \sqrt{1-x^2}$$

$$\text{From (ii), } \cos^{-1}x = \pi - \tan^{-1} \frac{\sqrt{1-y^2}}{y}$$

$$= \pi - \tan^{-1} \frac{\sqrt{1-x^2}}{-x}$$

$$= \pi + \tan^{-1} \frac{\sqrt{1-x^2}}{x}$$

$$= \pi + \cot^{-1} \frac{x}{\sqrt{1-x^2}} - \pi \quad (\text{as } x < 0)$$

$$= \cot^{-1} \frac{x}{\sqrt{1-x^2}}$$

8. (1), (3), (4)

$$(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^2$$

$$\therefore \sin^{-1}x + \sin^{-1}w = \sin^{-1}y + \sin^{-1}z = \pi$$

$$\text{or } \sin^{-1}x + \sin^{-1}w = \sin^{-1}y + \sin^{-1}z = -\pi$$

$$\therefore x = y = z = w = 1 \text{ or } x = y = z = w = -1$$

$$\text{Hence, the maximum value of } \begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \text{ and}$$

$$\text{minimum value } \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2.$$

Also, there are 16 different determinants as each place value is either 1 or -1.

9. (1), (2), (3), (4)

$$\text{Since } |\tan^{-1}x| = \begin{cases} \tan^{-1}x, & \text{if } x \geq 0 \\ -\tan^{-1}x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow |\tan^{-1}x| = \tan^{-1}|x| \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \tan|\tan^{-1}x| = \tan \tan^{-1}|x| = |x|$$

$$\text{Also, } |\cot^{-1}x| = \cot^{-1}|x| \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \cot|\cot^{-1}x| = x, \quad \forall x \in \mathbb{R}$$

$$\tan^{-1}|\tan x| = \begin{cases} x, & \text{if } 0 < x < \frac{\pi}{2} \\ -x, & \text{if } -\frac{\pi}{2} < x < 0 \end{cases}$$

$$\sin|\sin^{-1}x| = \begin{cases} x, & x \in [0, 1] \\ -x, & x \in [-1, 0) \end{cases}$$

10. (1), (3)

We have

$$\cot^{-1}\left(\frac{n^2 - 10n + 21.6}{\pi}\right) > \frac{\pi}{6}$$

$$\Rightarrow \frac{n^2 - 10n + 21.6}{\pi} < \cot \frac{\pi}{6}$$

(as $\cot x$ is decreasing for $0 < x < \pi$)

$$\text{or } n^2 - 10n + 21.6 < \pi\sqrt{3}$$

$$\text{or } n^2 - 10n + 25 + 21.6 - 25 < \pi\sqrt{3}$$

$$\text{or } (n-5)^2 < \pi\sqrt{3} + 3.4$$

$$\text{or } -\sqrt{\pi\sqrt{3} + 3.4} < n-5 < \sqrt{\pi\sqrt{3} + 3.4}$$

$$\text{or } 5 - \sqrt{\pi\sqrt{3} + 3.4} < n < 5 + \sqrt{\pi\sqrt{3} + 3.4} \quad \dots(i)$$

$$\text{Since } \sqrt{3}\pi = 5.5, \text{ nearly, } \sqrt{\pi\sqrt{3} + 3.4} \sim \sqrt{8.9} \sim 2.9$$

$$\Rightarrow 2.1 < n < 7.9$$

$$\therefore n = 3, 4, 5, 6, 7$$

{as $n \in \mathbb{N}$ }

11. (3), (4)

$$xy < 0$$

$$\Rightarrow x + \frac{1}{x} \geq 2, y + \frac{1}{y} \leq -2$$

$$\text{or } x + \frac{1}{x} \leq -2, y + \frac{1}{y} \geq 2$$

$$x + \frac{1}{x} \geq 2$$

$$\Rightarrow \sec^{-1}\left(x + \frac{1}{x}\right) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$$

$$y + \frac{1}{y} \leq -2$$

$$\Rightarrow \sec^{-1}\left(y + \frac{1}{y}\right) \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$$

$$\Rightarrow z \in \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$$

$$f(x) = \sin^{-1}|\sin kx| + \cos^{-1}(\cos kx)$$

$$\text{Let } g(x) = \sin^{-1}|\sin x| + \cos^{-1}(\cos x)$$

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi, & \frac{\pi}{2} < x \leq \frac{3\pi}{2} \\ 4\pi - 2x, & \frac{3\pi}{2} < x \leq 2\pi \end{cases}$$

$g(x)$ is periodic with period 2π and is constant in the continuous interval $\left[2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$ (where $n \in I$) and $f(x) = g(kx)$.

So, $f(x)$ is constant in the interval

$$\left[\frac{2n\pi}{k} + \frac{\pi}{2k}, \frac{2n\pi}{k} + \frac{3\pi}{2k}\right]$$

$$\text{Thus, } \frac{\pi}{4} = \frac{3\pi}{2k} - \frac{\pi}{2k}$$

$$\text{or } \frac{\pi}{k} = \frac{\pi}{4}$$

$$\text{or } k = 4$$

1), (2), (3), (4)

$$(1) \ y = \tan(\cos^{-1}x) \text{ and } y = \frac{\sqrt{1-x^2}}{x}$$

$$\text{or } y = \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \frac{\sqrt{1-x^2}}{x} \text{ and } y = \frac{\sqrt{1-x^2}}{x}$$

$$\therefore D_1 = [-1, 1] - \{0\} \text{ and } D_2 = [-1, 1] - \{0\}$$

So functions are identical and hence they have same graph.

$$(2) \ y = \tan(\cot^{-1}x) \text{ and } y = \frac{1}{x}$$

$$\therefore y = \tan\left(\tan^{-1}\frac{1}{x}\right) \text{ and } y = \frac{1}{x}$$

$$\text{or } y = \frac{1}{x} \text{ and } y = \frac{1}{x}$$

$$\therefore D_1 = R - \{0\} \text{ and } D_2 = R - \{0\}$$

So, functions are identical and hence they have same graph.

$$(3) \ y = \sin(\tan^{-1}x) \text{ and } y = \frac{x}{\sqrt{1+x^2}}$$

$$\text{or } y = \sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right) = \frac{x}{\sqrt{1+x^2}} \text{ and } y = \frac{x}{\sqrt{1+x^2}}$$

$$\text{or } D_1 = R, D_2 = R$$

So, functions are identical and hence they have same graph.

$$(4) \ y = \cos(\tan^{-1}x) \text{ and } y = \sin(\cot^{-1}x)$$

$$\text{or } y = \cos(\tan^{-1}x) = \cos\left[\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \frac{1}{\sqrt{1+x^2}}$$

$$\text{and } y = \sin(\cot^{-1}x) = \sin\left[\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore D_1 = R \text{ and } D_2 = R$$

So, functions are identical and hence they have same graph.

14. (1), (4)

For the given equation $0 \leq x, y \leq 1$.

$$\text{Also } \sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}x = \cos^{-1}y = \sin^{-1}\sqrt{1-y^2}$$

$$\Rightarrow x = \sqrt{1-y^2}$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\text{Again, } \sin 2x = \cos 2y$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - 2x\right) = \cos 2y$$

$$\Rightarrow \frac{\pi}{2} - 2x = 2n\pi \pm 2y, \text{ where } n \in I$$

$$\Rightarrow x \pm y = \frac{\pi}{4} \quad (\because x, y \in [-1, 1]) \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$x = \frac{\pi}{8} + \sqrt{\frac{1}{2} - \frac{\pi^2}{64}}$$

$$\text{and } y = \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} - \frac{\pi}{8}$$

15. (1), (2)

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$

$$\text{or } \sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi/2$$

$$\text{or } \cos^{-1}x + \cos^{-1}y = \cos^{-1}(-z)$$

$$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$\text{or } x^2 + y^2 + z^2 + 2xyz = 1$$

16. (1), (3)

The given relation is possible when

$$a - \frac{a^2}{3} + \frac{a^3}{9} - \dots = 1 + b + b^2 + \dots$$

$$\text{Also, } -1 \leq a - \frac{a^2}{3} + \frac{a^3}{9} - \dots \leq 1$$

$$\text{and } -1 \leq 1 + b + b^2 + \dots \leq 1$$

$$\Rightarrow |b| < 1 \text{ and } |a| < 3 \text{ and } \frac{a}{1+\frac{a}{3}} = \frac{1}{1-b}$$

$$\text{or } \frac{3a}{a+3} = \frac{1}{1-b}$$

$$\text{or } 3a - 3ab = a + 3$$

$$\text{or } 2a - 3ab = 3$$

$$\text{or } b = \frac{2a-3}{3a} \text{ and } a = \frac{3}{2-3b}$$

17. (1), (2)

$$\tan^{-1}(x^2 + 3|x| - 4) + \cot^{-1}(4\pi + \sin^{-1}\sin 14) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}(x^2 + 3|x| - 4) + \cot^{-1}(4\pi + 14 - 4\pi) = \frac{\pi}{2}$$

$$\Rightarrow x^2 + 3|x| - 4 = 14$$

$$\Rightarrow x^2 + 6|x| - 3|x| - 18 = 0$$

$$\Rightarrow (|x| + 6)(|x| - 3) = 0$$

$$\Rightarrow x = \pm 3$$

$$\text{So, } \sin^{-1}\sin 2x = \sin^{-1}\sin 6 = 6 - 2\pi$$

$$\text{or } \sin^{-1}\sin(-6) = 2\pi - 6.$$

18. (1), (2)

We know that

$$\text{if } |x| \leq 1, \text{ then } 2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

$$\text{if } x > 1, \text{ then } 2 \tan^{-1} x = \pi - \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

$$\text{if } x < -1, \text{ then } 2 \tan^{-1} x = -\pi - \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

Hence, the required values are $x < -1$ or $x > 1$.

19. (1), (2)

Put $x = \tan \theta$

$$\Rightarrow \alpha = \tan^{-1}(\tan 4\theta)$$

$$= 4\theta - \pi \text{ for } x \in \left[1, \frac{1}{k}\right)$$

$$= 4\theta \text{ for } x \in (-k, k)$$

$$\text{Also, } \beta = 2(\pi - 2\theta) \text{ for } x \in \left[1, \frac{1}{k}\right)$$

$$= 4\theta \text{ for } x \in (-k, k)$$

20. (1), (2), (3)

Let $\tan^{-1} x = \alpha$ and $\tan^{-1} x^3 = \beta$ $\tan \alpha = x$ and $\tan \beta = x^3$

$$\therefore 2 \tan(\alpha + \beta) = \frac{2(\tan \alpha + \tan \beta)}{1 - \tan \alpha \tan \beta} = 2 \left[\frac{x + x^3}{1 - x^4} \right] = \frac{2x}{1 - x^2}$$

$$\begin{aligned} \text{Also } \frac{2x}{1 - x^2} &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \tan 2\alpha = \tan(2 \tan^{-1} x) \\ &= \tan \left(2 \left(\frac{\pi}{2} - \cot^{-1} x \right) \right) \\ &= \tan(\pi - \cot^{-1} x - \cot^{-1} x) \\ &= \tan(\cot^{-1}(-x) - \cot^{-1}(x)) \end{aligned}$$

21. (1), (2)

$$\alpha = \sin^{-1} \frac{36}{85} \Rightarrow \sin \alpha = \frac{36}{85} \Rightarrow \tan \alpha = \frac{36}{77}$$

$$\beta = \cos^{-1} \left(\frac{4}{5} \right) \Rightarrow \cos \beta = \frac{4}{5} \Rightarrow \tan \beta = \frac{3}{4}$$

$$\text{and } \tan \gamma = \frac{8}{15}$$

$$\begin{aligned} \therefore \tan(\alpha + \beta + \gamma) &= \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha} \\ &= \frac{\frac{36}{77} + \frac{3}{4} + \frac{8}{15} - \frac{36}{77} \cdot \frac{3}{4} \cdot \frac{8}{15}}{1 - \left(\frac{36}{77} \times \frac{3}{4} + \frac{8}{15} \times \frac{3}{4} + \frac{8}{15} \times \frac{36}{77} \right)} \end{aligned}$$

$$\Rightarrow \tan(\alpha + \beta + \gamma) = \infty$$

$$\Rightarrow \alpha + \beta + \gamma = \frac{\pi}{2}$$

$$\Rightarrow \cot \alpha + \cot \beta + \cot \gamma = \cot \alpha \cot \beta \cot \gamma$$

$$\Rightarrow \tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \alpha \tan \gamma = 1$$

22. (1), (2), (4)

Let t_r denote the r th term of the series 3, 7, 13, 21, ... and

$$S = 3 + 7 + 13 + 21 + \dots + t_n$$

$$S = 3 + 7 + 13 + \dots + t_{n-1} + t_n$$

$$\begin{array}{r} S \\ - S \\ \hline 0 = 3 + 4 + 6 + 8 + \dots + 2n - t_n \end{array}$$

$$\Rightarrow t_n = 3 + 4 + 6 + \dots + 2n = 1 + 2 \times \frac{1}{2} n(n+1) = n^2 + n + 1$$

$$\text{Let } T_r = \cot^{-1}(r^2 + r + 1)$$

$$\begin{aligned} &= \tan^{-1} \left(\frac{1}{r^2 + r + 1} \right) = \tan^{-1} \left(\frac{r+1-r}{1+r(r+1)} \right) \\ &= \tan^{-1}(r+1) - \tan^{-1} r \end{aligned}$$

Thus, the sum of the first n terms of the given series is

$$\begin{aligned} \sum_{r=1}^n [\tan^{-1}(r+1) - \tan^{-1} r] &= \tan^{-1}(n+1) - \tan^{-1}(1) \\ &= \tan^{-1} \left[\frac{n+1-1}{1+1(n+1)} \right] \\ &= \tan^{-1} \left(\frac{n}{n+2} \right) \\ &= \tan^{-1} \left(\frac{1}{1+\frac{2}{n}} \right) \end{aligned}$$

$$\Rightarrow S_{\infty} = \lim_{n \rightarrow \infty} \tan^{-1} \left(\frac{1}{1+\frac{2}{n}} \right) = \frac{\pi}{4}$$

$$S_{10} = \tan^{-1} \frac{10}{12} = \tan^{-1} \frac{5}{6}$$

$$S_6 = \tan^{-1} \frac{3}{4} = \sin^{-1} \frac{3}{5}$$

$$S_{20} = \tan^{-1} \frac{10}{11} = \cot^{-1} 1.1$$

23. (1), (3)

Given equation is $x^2 + 2x \sin(\cos^{-1} y) + 1 = 0$.Since x is real, $D \geq 0$. Therefore,

$$4(\sin(\cos^{-1} y))^2 - 4 \geq 0$$

$$\text{or } (\sin(\cos^{-1} y))^2 \geq 1$$

$$\text{or } \sin(\cos^{-1} y) = \pm 1$$

$$\text{or } \cos^{-1} y = \frac{\pi}{2} \Rightarrow y = 0$$

Putting value of y in the original equation, we have

$$x^2 + 2x + 1 = 0 \Rightarrow x = -1.$$

Hence, the equation has only one solution.

24. (2), (3)

$$1 \leq \frac{\pi}{\cos^{-1} x} < \infty$$

$$\Rightarrow 2 \leq \frac{\pi}{\cos^{-1} x} < \infty$$

Hence, 2 should lie between or on the roots of

$$t^2 - \left(a + \frac{1}{2}\right)t - a^2 = 0, \text{ where } t = 2\pi/\cos^{-1} x.$$

$$\Rightarrow f(2) \leq 0 \Rightarrow a^2 + 2a - 3 \geq 0$$

$$\Rightarrow a \in (-\infty, -3] \cup [1, \infty)$$

(4), 2. (2), 3. (3)

$$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1} y \in [0, \pi]$$

$$\sec^{-1} z \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

$$\Rightarrow \sin^{-1} x + \cos^{-1} y + \sec^{-1} z \leq \frac{\pi}{2} + \pi + \pi = \frac{5\pi}{2}$$

$$\text{Also, } t^2 - \sqrt{2\pi} t + 3\pi = t^2 - 2\sqrt{\frac{\pi}{2}} t + \frac{\pi}{2} - \frac{\pi}{2} + 3\pi$$

$$= \left(t - \sqrt{\frac{\pi}{2}}\right)^2 + \frac{5\pi}{2} \geq \frac{5\pi}{2}$$

The given inequation exists if equality holds, i.e.,

$$\text{L.H.S.} = \text{R.H.S.} = \frac{5\pi}{2}$$

$$\Rightarrow x = 1, y = -1, z = -1 \text{ and } t = \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \cos^{-1}(\cos 5t^2) = \cos^{-1}\left(\cos\left(\frac{5\pi}{2}\right)\right) = \frac{\pi}{2}$$

$$\cos^{-1}(\min\{x, y, z\}) = \cos^{-1}(-1) = \pi$$

(2), 5. (1), 6. (2)

Given $ax + b(\sec(\tan^{-1} x)) = c$ and $ay + b(\sec(\tan^{-1} y)) = c$

Let $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$, then the given relations are

$$a \tan \alpha + b \sec \alpha = c \text{ and } a \tan \beta + b \sec \beta = c$$

From these two relations, we can conclude that equation $a \tan \theta + b \sec \theta = c$ has roots α and β .

$$a \tan \theta + b \sec \theta = c$$

$$\text{or } b \sec \theta = c - a \tan \theta$$

$$\text{or } b^2 \sec^2 \theta = c^2 - 2ac \tan \theta + a^2 \tan^2 \theta$$

$$\text{or } b^2 + b^2 \tan^2 \theta = c^2 - 2ac \tan \theta + a^2 \tan^2 \theta$$

$$\text{or } (a^2 - b^2) \tan^2 \theta - 2ac \tan \theta + c^2 - b^2 = 0$$

$$\text{Therefore, sum of the roots, } \tan \alpha + \tan \beta = x + y = \frac{2ac}{a^2 - b^2}$$

$$\text{and the product of roots, } \tan \alpha \tan \beta = xy = \frac{c^2 - b^2}{a^2 - b^2}$$

$$\text{and } \frac{x+y}{1-xy} = \frac{\frac{2ac}{a^2-b^2}}{1-\frac{c^2-b^2}{a^2-b^2}} = \frac{2ac}{a^2-c^2}$$

7. (2), 8. (4), 9. (3)

Let $\cos^{-1} x = a \Rightarrow a \in [0, \pi]$

and $\sin^{-1} y = b \Rightarrow b \in [-\pi/2, \pi/2]$

$$\text{We have } a + b^2 = \frac{p\pi^2}{4}$$

...(i)

$$\text{and } ab^2 = \frac{\pi^4}{16}$$

...(ii)

Since $b^2 \in [0, \pi^2/4]$, we get $a + b^2 \in [0, \pi + \pi^2/4]$

So, from Eq. (i) we get $0 \leq \frac{p\pi^2}{4} \leq \pi + \frac{\pi^2}{4}$

$$\text{i.e., } 0 \leq p \leq \frac{4}{\pi} + 1$$

Since $p \in \mathbb{Z}$, so $p = 0, 1$ or 2 .

Substituting the value of b^2 from Eq. (i) in Eq. (ii), we get

$$a \left(\frac{p\pi^2}{4} - a \right) = \frac{\pi^4}{16}$$

$$\Rightarrow 16a^2 - 4p\pi^2 a + \pi^4 = 0$$

Since $a \in \mathbb{R}$, we have $D \geq 0$

$$\text{i.e., } 16p^2\pi^4 - 64\pi^4 \geq 0$$

$$\text{or } p^2 \geq 4 \text{ or } p \geq 2 \text{ or } p = 2$$

Substituting $p = 2$ in Eq. (iii), we get

$$16a^2 - 8\pi^2 a + \pi^4 = 0$$

$$\text{or } (4a - \pi^2)^2 = 0$$

$$\text{or } a = \frac{\pi^2}{4} = \cos^{-1} x \text{ or } x = \cos \frac{\pi^2}{4}$$

$$\text{From Eq. (ii), we get } \frac{\pi^2}{4} b^2 = \frac{\pi^4}{16}$$

$$\text{or } b = \pm \frac{\pi}{2} = \sin^{-1} y \text{ or } y = \pm 1$$

10. (3), 11. (1), 12. (4)

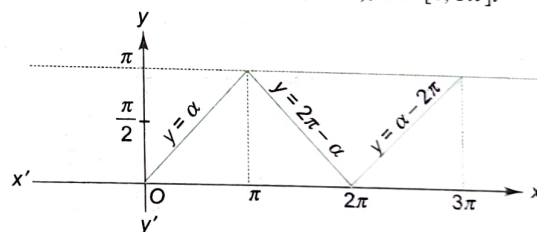
Let $\cos^{-1} x = \theta$ or $x = \cos \theta$, where $\theta \in [0, \pi]$

$$\cos^{-1}(4x^3 - 3x) = \cos^{-1}(4\cos^3 \theta - 3\cos \theta)$$

$$= \cos^{-1}(\cos 3\theta) = \cos^{-1}(\cos \alpha)$$

where $\alpha = 3\theta \in [0, 3\pi]$

Refer to the graph of $y = \cos^{-1}(\cos \alpha)$, $\alpha \in [0, 3\pi]$.



From the graph,

$$\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos \alpha)$$

$$= \begin{cases} \alpha, & 0 \leq \alpha < \pi \\ 2\pi - \alpha, & \pi \leq \alpha \leq 2\pi \\ \alpha - 2\pi, & 2\pi < \alpha \leq 3\pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & 0 \leq 3\cos^{-1}x < \pi \\ 2\pi - 3\cos^{-1}x, & \pi \leq 3\cos^{-1}x \leq 2\pi \\ 3\cos^{-1}x - 2\pi, & 2\pi < 3\cos^{-1}x \leq 3\pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & 0 \leq \cos^{-1}x < (\pi/3) \\ 2\pi - 3\cos^{-1}x, & (\pi/3) \leq \cos^{-1}x \leq (2\pi/3) \\ 3\cos^{-1}x - 2\pi, & (2\pi/3) < \cos^{-1}x \leq \pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & (1/2) < x \leq 1 \\ 2\pi - 3\cos^{-1}x, & (-1/2) \leq x \leq (1/2) \\ 3\cos^{-1}x - 2\pi, & -1 \leq x < (-1/2) \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \leq x < (-1/2) \\ 2\pi - 3\cos^{-1}x, & (-1/2) \leq x \leq (1/2) \\ 3\cos^{-1}x, & (1/2) < x \leq 1 \end{cases}$$

$$\text{For } x \in \left[-1, -\frac{1}{2}\right], \cos^{-1}(4x^3 - 3x) = 3 \cos^{-1} x - 2\pi$$

$$\Rightarrow a = -2\pi \text{ and } b = 3 \Rightarrow a + b\pi = \pi$$

$$\text{For } x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \cos^{-1}(4x^3 - 3x) = 2\pi - 3 \cos^{-1} x$$

$$\Rightarrow a = 2\pi \text{ and } b = -3$$

$$\begin{aligned} \Rightarrow \sin^{-1}\left(\sin \frac{a}{b}\right) &= \sin^{-1}\left(\sin \frac{2\pi}{-3}\right) \\ &= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \end{aligned}$$

$$\text{For } x \in \left(\frac{1}{2}, 1\right], \cos^{-1}(4x^3 - 3x) = 3 \cos^{-1} x$$

$$\Rightarrow a = 0 \text{ and } b = 3$$

$$\therefore \lim_{y \rightarrow a} b \cos(y) = \lim_{y \rightarrow 0} 3 \cos(y) = 3$$

13. (3) $a = 20 - 6\pi, b = 10\pi - 30, c = \sin^{-1} \sin(4\pi - 10) = 10 - 3\pi$
So, $a + b + c = \pi$

$$\sin^{-1} \sin x \geq |x - \pi| \Rightarrow x \in \left[\frac{\pi}{2}, \pi\right]$$

So, largest integer $x = 3$.

14. (2) $5 \sec^{-1} x + 10 \sin^{-1} y = 10\pi$

$$\Rightarrow \sec^{-1} x = \pi \text{ and } \sin^{-1} y = \frac{\pi}{2}$$

$$\Rightarrow x = -1, y = 1$$

$$\text{So, } \tan^{-1}(-1) + \cos^{-1}(0) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$$

15. (2), 16. (1)

We have $f(x) = \sin^{-1} x$, having range $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Consider the function $h(x) = (\pi - \sin^{-1} x)$, where

$$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Clearly, functions $f(x)$ and $h(x)$ are identical.

Advantage with function $y = h(x)$ is that we are well versed

with $\sin^{-1} x$ if $(\sin^{-1} x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned} h(\sin 10) &= \pi - \sin^{-1}(\sin 10) \\ &= \pi - \sin^{-1}(\sin(3\pi - 10)) \\ &= \pi - (3\pi - 10) \\ &= 10 - 2\pi \end{aligned}$$

$$\text{Now, } f(x) < \frac{3\pi}{4}$$

$$\therefore h(x) < \frac{3\pi}{4}$$

$$\Rightarrow \pi - \sin^{-1} x < \frac{3\pi}{4}$$

$$\Rightarrow \sin^{-1} x > \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} < \sin^{-1} x \leq \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}} < x \leq 1$$

Matrix Match Type

1. $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r$

a. $\sin^{-1} \frac{4}{5} = \tan^{-1} \frac{4}{3}$

$$2 \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{3}{4} = \cot^{-1} \frac{4}{3}$$

$$\text{and } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

b. $\tan^{-1} \frac{12}{5} + \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{63}{16}$
 $= \pi + \tan^{-1} \frac{48+15}{20-36} + \tan^{-1} \frac{63}{16}$
 $= \pi - \tan^{-1} \frac{63}{16} + \tan^{-1} \frac{63}{16} = \pi$

c. $A = \tan^{-1} \frac{x\sqrt{3}}{2\lambda - x}$ and $B = \tan^{-1} \left(\frac{2x - \lambda}{\lambda\sqrt{3}} \right)$

$$\text{Now, } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\begin{aligned} &= \frac{\frac{x\sqrt{3}}{2\lambda - x} - \frac{2x - \lambda}{\lambda\sqrt{3}}}{1 + \frac{x\sqrt{3}}{2\lambda - x} \cdot \frac{2x - \lambda}{\lambda\sqrt{3}}} \\ &= \frac{3x\lambda + (2x - \lambda)(x - 2\lambda)}{\sqrt{3} [\lambda(2\lambda - x) + x(2x - \lambda)]} \\ &= \frac{1}{\sqrt{3}} \left[\frac{2x^2 - 2\lambda x + 2\lambda^2}{2x^2 - 2\lambda x + 2\lambda^2} \right] \\ &= \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6} \end{aligned}$$

$$\therefore A - B = \frac{\pi}{6}$$

d. $\tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{2 \cdot \frac{1}{3}}{1 - \frac{1}{9}}$

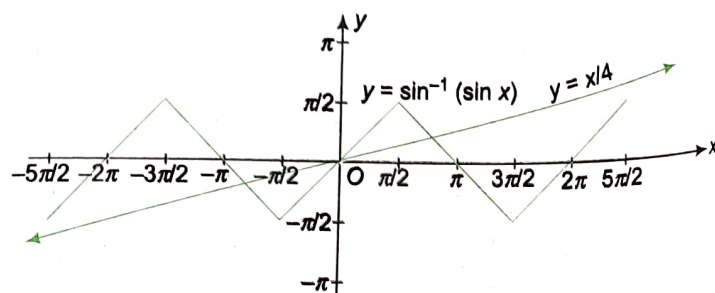
$$= \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{3}{4}$$

$$= \tan^{-1} \frac{1 + \frac{3}{4}}{1 - \frac{1}{7} \cdot \frac{3}{4}}$$

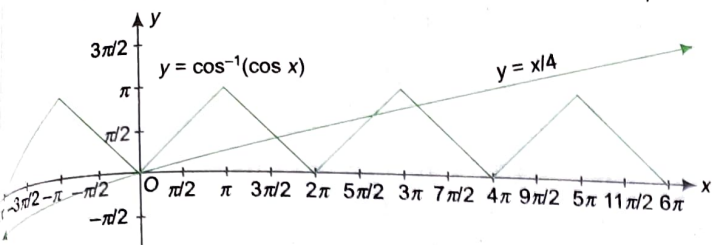
$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

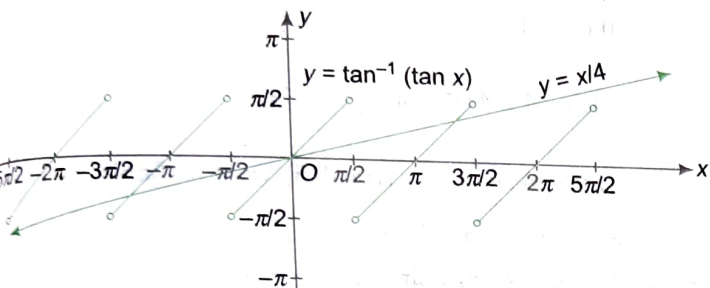
2. (4) a. Graph of $y = \sin^{-1}(\sin x)$ and $y = x/4$ intersect at three points. So, number of solutions is 3.



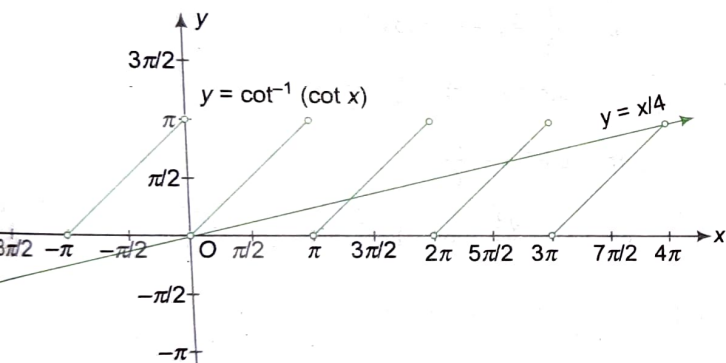
- b. Graph of $y = \cos^{-1}(\cos x)$ and $y = x/4$ intersect at four points. So, number of solutions is 4.



- c. Graph of $y = \tan^{-1}(\tan x)$ and $y = x/4$ intersect at three points. So, number of solutions is 3.



- d. Graph of $y = \cot^{-1}(\cot x)$ and $y = x/4$ intersect at two points. So, number of solutions is 2.



3. (2) a. $\sin^{-1}(3x - 4x^3) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \pi - 3\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow 3\sin^{-1}x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$\Rightarrow \sin^{-1}x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$$

$$\Rightarrow x \in \left[\frac{1}{2}, 1\right]$$

b. $\sin^{-1} \frac{2x}{1+x^2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \pi - 2\tan^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow 2\tan^{-1}x \in \left[\frac{\pi}{2}, \pi\right]$$

$$\Rightarrow \tan^{-1}x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\Rightarrow x \in [1, \infty)$$

c. $\tan^{-1} \frac{2x}{1-x^2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow 2\tan^{-1}x - \pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow 2\tan^{-1}x \in \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \tan^{-1}x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\Rightarrow x \in (1, \infty)$$

d. $\sin^{-1}(2x\sqrt{1-x^2}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \pi - 2\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow 2\sin^{-1}x \in \left[\frac{\pi}{2}, \pi\right]$$

$$\Rightarrow \sin^{-1}x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

$$\Rightarrow x \in \left[\frac{1}{\sqrt{2}}, 1\right]$$

4. (3) a. $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 = \frac{\pi^2}{2}$

$$\Rightarrow (\sin^{-1}x)^2 = (\sin^{-1}y)^2 = \frac{\pi^2}{4}$$

$$\Rightarrow \sin^{-1}x = \pm \frac{\pi}{2}, \sin^{-1}y = \pm \frac{\pi}{2}$$

$$\Rightarrow x = \pm 1 \text{ and } y = \pm 1$$

$$\Rightarrow x^3 + y^3 = -2, 0, 2$$

b. $(\cos^{-1}x)^2 + (\cos^{-1}y)^2 = 2\pi^2$

$$\Rightarrow (\cos^{-1}x)^2 = (\cos^{-1}y)^2 = \pi^2$$

$$\Rightarrow x = y = -1$$

$$\Rightarrow x^5 + y^5 = -1$$

c. $(\sin^{-1}x)^2 (\cos^{-1}y)^2 = \frac{\pi^4}{4}$

$$\Rightarrow (\sin^{-1}x)^2 = \frac{\pi^2}{4} \text{ and } (\cos^{-1}y)^2 = \pi^2$$

$$\Rightarrow (\sin^{-1}x) = \pm \frac{\pi}{2} \text{ and } (\cos^{-1}y) = \pi$$

$$\Rightarrow x = \pm 1 \text{ and } y = -1$$

$$x - y = 0, 2$$

d. $|\sin^{-1}x - \sin^{-1}y| = \pi$

$$\Rightarrow \sin^{-1}x = -\frac{\pi}{2} \text{ and } \sin^{-1}y = \frac{\pi}{2}$$

$$\Rightarrow x = -1 \text{ and } y = 1$$

$$\text{or } \sin^{-1}x = \frac{\pi}{2} \text{ and } \sin^{-1}y = -\frac{\pi}{2}$$

$$\Rightarrow x = 1 \text{ and } y = -1$$

$$\Rightarrow x - y = -2, 2$$

5. $a \rightarrow s; b \rightarrow p; c \rightarrow q; d \rightarrow r$

a. $f(x) = \sin^{-1}x + \cos^{-1}x + \cot^{-1}x = \frac{\pi}{2} + \cot^{-1}x, x \in [-1, 1]$

For $x \in [-1, 1], \cot^{-1}x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$\Rightarrow \frac{\pi}{2} + \cot^{-1}x \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

b. $f(x) = \cot^{-1}x + \tan^{-1}x + \operatorname{cosec}^{-1}x$

$= \frac{\pi}{2} + \operatorname{cosec}^{-1}x, \text{ where } x \in (-\infty, -1] \cup [1, \infty)$

Now $\operatorname{cosec}^{-1}x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$\Rightarrow \frac{\pi}{2} + \operatorname{cosec}^{-1}x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

c. $f(x) = \cot^{-1}x + \tan^{-1}x + \cos^{-1}x$

$= \frac{\pi}{2} + \cos^{-1}x, \text{ where } x \in [-1, 1]$

$\Rightarrow \frac{\pi}{2} + \cos^{-1}x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

d. $f(x) = \sec^{-1}x + \operatorname{cosec}^{-1}x + \sin^{-1}x, \text{ where } x \in \{-1, 1\}$

$= \frac{\pi}{2} + \sin^{-1}x, \text{ where } x \in \{-1, 1\}$

Hence, $f(x) \in \{0, \pi\}$.

Numerical Value Type

1. (5) $(\cot^{-1}x)(\tan^{-1}x) + \left(2 - \frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2 - \frac{\pi}{2}\right) > 0$

$\Rightarrow \cot^{-1}x \left(\tan^{-1}x - \frac{\pi}{2}\right) + 2\cot^{-1}x - 6 - 3\left(\tan^{-1}x - \frac{\pi}{2}\right) > 0$

$\Rightarrow -(\cot^{-1}x)^2 + 5\cot^{-1}x - 6 > 0$

$\Rightarrow (\cot^{-1}x - 3)(2 - \cot^{-1}x) > 0$

$\Rightarrow (\cot^{-1}x - 3)(\cot^{-1}x - 2) < 0$

$\Rightarrow 2 < \cot^{-1}x < 3$

$\Rightarrow \cot 3 < x < \cot 2$ [as $\cot^{-1}x$ is a decreasing function]

$\Rightarrow \text{Hence, } x \in (\cot 3, \cot 2)$

$\Rightarrow \cot^{-1}a + \cot^{-1}b = \cot^{-1}(\cot 3) + \cot^{-1}(\cot 2) = 5$

2. (2) Since $\sin^{-1}$ is defined for $[-1, 1]$, so we have $a = 0$

$\therefore x = \sin^{-1}1 + \cos^{-1}1 - \tan^{-1}1 = \frac{\pi}{4}$

$\Rightarrow \sec^2 x = 2$

3. (6) Let $\tan^{-1}u = \alpha$ or $\tan \alpha = u$

$\tan^{-1}v = \beta$ or $\tan \beta = v$

$\tan^{-1}w = \gamma$ or $\tan \gamma = w$

$\tan(\alpha + \beta + \gamma) = \frac{s_1 + s_2 + s_3}{1 - s_2} = \frac{0 - (-11)}{1 - (-10)} = \frac{11}{11} = 1$

$\therefore \alpha + \beta + \gamma = \tan^{-1}(1) = \frac{\pi}{4}$

$\Rightarrow 3\operatorname{cosec}^2(\tan^{-1}u + \tan^{-1}v + \tan^{-1}w) = 6$

4. (3) $\sin^{-1}\left(x^2 - \frac{x^4}{3} + \frac{x^6}{9} - \dots\right) + \cos^{-1}\left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9} - \dots\right) = \frac{\pi}{2}$

$\Rightarrow \left(x^2 - \frac{x^4}{3} + \frac{x^6}{9} - \dots\right) = \left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9} - \dots\right)$

or $\frac{x^2}{1 + \frac{x^2}{3}} = \frac{x^4}{1 + \frac{x^4}{3}}$

or $\frac{3}{3 + x^2} = \frac{3x^2}{3 + x^4}$ or $x = 0$

or $9 + 3x^4 = 9x^2 + 3x^4$ or $x = 0$

or $x^2 = 1 \Rightarrow x = 0, 1$ or -1

Therefore, the number of values is equal to 3.

5. (7) $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$ is defined

If $\cos^{-1}4x \geq \frac{\pi}{3} \Rightarrow 4x \leq \frac{1}{2}$ or $x \leq \frac{1}{8}$... (i)

Also, $-1 \leq 4x \leq 1$ or $-\frac{1}{4} \leq x \leq \frac{1}{4}$... (ii)

Therefore, from Eqs. (i) and (ii), we have domain: $x \in \left[-\frac{1}{4}, \frac{1}{8}\right]$

$\Rightarrow 4a + 64b = 7$

6. (9) $1 + \sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + \dots = 2$

$\Rightarrow \frac{1}{1 - \sin(\cos^{-1}x)} = 2$

or $\frac{1}{2} = 1 - \sin(\cos^{-1}x)$

or $\sin(\cos^{-1}x) = \frac{1}{2}$

or $\cos^{-1}x = \frac{\pi}{6}$

or $x = \frac{\sqrt{3}}{2}$ or $12x^2 = 9$

7. (9) $\tan^{-1}\left(x + \frac{3}{x}\right) - \tan^{-1}\left(x - \frac{3}{x}\right) = \tan^{-1}\frac{6}{x}$

or $\tan^{-1}\left(\frac{\left(x + \frac{3}{x}\right) - \left(x - \frac{3}{x}\right)}{1 + \left(x + \frac{3}{x}\right)\left(x - \frac{3}{x}\right)}\right) = \tan^{-1}\frac{6}{x}$

$\Rightarrow x^2 - \frac{9}{x^2} = 0$ or $x^4 = 9$

8. (4) $f(x) = \sin^{-1}x + 2\tan^{-1}x + (x + 2)^2 - 3$

Domain of $f(x)$ is $[-1, 1]$.

Also $f(x)$ is an increasing function in the domain. Therefore,

$p = f_{\min}(x) = f(-1) = -\frac{\pi}{2} + 2\left(\frac{-\pi}{4}\right) + 1 - 3 = -\pi - 2$

and $q = f_{\max}(x) = f(1) = \frac{\pi}{2} + 2\left(\frac{\pi}{4}\right) + 9 - 3 = \pi + 6$.

Therefore, the range of $f(x)$ is $[-\pi - 2, \pi + 6]$.

Hence,

9. (9) $\sec^2 u, \sec^4 v, \sec^6 w \in [1, \infty)$

$\therefore \sec^2(u) + \sec^4(v) + \sec^6(w) \in [3, \infty)$

$\therefore \pi(\sec^2 u + \sec^4 v + \sec^6 w) \in [3\pi, \infty)$

But $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z \in [0, 3\pi]$

So equation is possible if L.H.S. = R.H.S. = 3π

$\therefore \cos^{-1}x = \cos^{-1}y = \cos^{-1}z = \pi$

$\therefore x = y = z = -1$

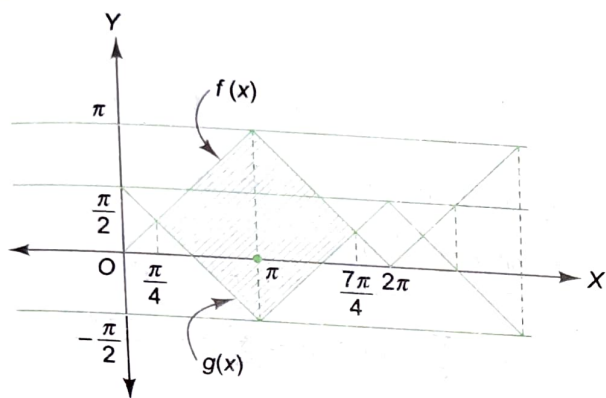
$$\text{and } \sec^2 u = \sec^4 v = \sec^6 w = 1$$

$$\therefore u = \pi, v = 2\pi, w = 3\pi$$

$$\therefore x^{2000} + y^{2002} + z^{2004} + \frac{36\pi}{u+v+w} = 1 + 1 + 1 + \frac{36\pi}{6\pi} = 9$$

$$(8) \text{ We have } g(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$$

Both the curves bound the regions of same area in $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$, $\left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ and so on.



$\therefore$ Required area = Area of shaded square

$$= \frac{9\pi^2}{8} = \frac{a\pi^2}{b}$$

$$\therefore a = 9 \text{ and } b = 8$$

$$1. (-3) \text{ We must have } x(x+3) \geq 0$$

$$\Rightarrow x \geq 0 \text{ or } x \leq -3$$

$$\text{Also, } -1 \leq x^2 + 3x + 1 \leq 1$$

$$\Rightarrow x(x+3) \leq 0 \Rightarrow -3 \leq x \leq 0$$

$$\text{From Eqs. (i) and (ii), we get } x = \{0, -3\}.$$

$$\text{Also, for these values } x^2 + 3x + 1 \geq -1$$

Hence, the required sum is -3.

$$12. (1) \text{ Given expression is defined only for } x = 1 \text{ and } -1. \text{ Therefore,}$$

$$f(1) = 1 \text{ and } f(-1) = (1 + \pi)(1 + \pi) = (1 + \pi)^2$$

Hence, the least value is 1.

$$13. (25) \cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$$

$$\text{or } \cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x) = \cos^{-1}(-x)$$

$$\text{or } \cos^{-1}[(2x)(3x) - \sqrt{1-4x^2}\sqrt{1-9x^2}] = \cos^{-1}(-x)$$

$$\text{or } 6x^2 - \sqrt{1-4x^2}\sqrt{1-9x^2} = -x$$

$$\text{or } (6x^2 + x)^2 = (1-4x^2)(1-9x^2)$$

$$\text{or } x^2 + 12x^3 = 1 - 13x^2$$

$$\text{or } 12x^3 + 14x^2 - 1 = 0$$

$$\Rightarrow a = 12; b = 14; c = 0$$

$$\Rightarrow a + b + c = 12 + 14 - 1 = 25$$

$$14. (1) \tan^{-1}(3x) + \tan^{-1}(5x) = \tan^{-1}(7x) + \tan^{-1}(2x)$$

$$\text{or } \tan^{-1}(3x) - \tan^{-1}(2x) = \tan^{-1}(7x) - \tan^{-1}(5x)$$

$$\text{or } \tan^{-1}\left(\frac{3x-2x}{1+6x^2}\right) = \tan^{-1}\left(\frac{7x-5x}{1+35x^2}\right)$$

$$\text{or } \frac{x}{1+6x^2} = \frac{2x}{1+35x^2}$$

$$\Rightarrow x = 0 \text{ or } 1 + 35x^2 = 2 + 12x^2$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{\sqrt{23}} \text{ or } -\frac{1}{\sqrt{23}}$$

$$15. (0) \text{ Now, } \sin(\cos^{-1}(\tan(\sec^{-1}x)))$$

$$= \sin(\cos^{-1}(\tan(\tan^{-1}\sqrt{x^2-1})))$$

$$= \sin(\cos^{-1}(\sqrt{x^2-1}))$$

$$= \sin(\sin^{-1}(\sqrt{2-x^2}))$$

$$= \sqrt{2-x^2}$$

For $x \in [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$ and $x \geq -1$, we have

$$2 - x^2 = 1 + x$$

$$\therefore x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2} = 0.615, -1.615$$

None of which satisfies the condition $x \in [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.

$$16. (1) \sin^{-1}(x^2 + x + 1) + \cos^{-1}(\lambda x + 1) = \frac{\pi}{2}$$

$$\Rightarrow x^2 + x + 1 = \lambda x + 1$$

$$\Rightarrow x^2 + (1-\lambda)x = 0$$

$$\Rightarrow x = 0 \text{ or } x = \lambda - 1$$

Also, $\sin^{-1}(x^2 + x + 1)$ will exist if $-1 \leq x^2 + x + 1 \leq 1$.

$$\Rightarrow x^2 + x \leq 0 \Rightarrow -1 \leq x < 0$$

For exactly two solution, we have

$$-1 \leq \lambda - 1 < 0$$

$$\Rightarrow 0 \leq \lambda < 1$$

$$\therefore a = 0 \text{ and } b = 1$$

$$17. (8) \text{ Let } \alpha = \sin^{-1}\frac{\sqrt{5}}{3}, \text{ then } \cos^{-1}\frac{\sqrt{5}}{3} = \frac{\pi}{2} - \alpha$$

Hence the expression becomes

$$\begin{aligned} \sin\left\{2\left(\alpha - \frac{\pi}{2} + \alpha\right)\right\} &= \sin(4\alpha - \pi) \\ &= -\sin 4\alpha \\ &= -4 \sin \alpha \cos \alpha (1 - 2 \sin^2 \alpha) \\ &= (-4) \left(\frac{\sqrt{5}}{3}\right) \left(\frac{2}{3}\right) \left(\frac{-1}{9}\right) \\ &= \frac{8\sqrt{5}}{81} \end{aligned}$$

$$18. (3) \cos(2\sin^{-1}(\cot(\tan^{-1}(\sec(6\operatorname{cosec}^{-1}x)))) = -1$$

$$\Rightarrow \sin^{-1}(\cot(\tan^{-1}(\sec(6\operatorname{cosec}^{-1}x)))) = \pm \frac{\pi}{2}$$

$$\Rightarrow \cot(\tan^{-1}(\sec(6\operatorname{cosec}^{-1}x))) = \pm 1$$

$$\Rightarrow \tan^{-1}(\sec(6\operatorname{cosec}^{-1}x)) = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\Rightarrow \sec(6\operatorname{cosec}^{-1}x) = \pm 1$$

$$\Rightarrow 6\operatorname{cosec}^{-1}x = \pm 3\pi, \pm 2\pi, \pm \pi$$

$$\Rightarrow \operatorname{cosec}^{-1}x = \pm \frac{\pi}{2}, \pm \frac{\pi}{3}, \pm \frac{\pi}{6}$$

$$\Rightarrow x = 1, \frac{2}{\sqrt{3}}, 2$$

($\because x > 0$)

JEE MAIN

Single Correct Answer Type

1. (1) $2y = x + z$

As $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in A.P., we have

$$2 \tan^{-1} y = \tan^{-1} x + \tan^{-1} z$$

$$\Rightarrow 2 \tan^{-1} y = \tan^{-1} \frac{x+z}{1-xz}$$

$$\Rightarrow \tan^{-1} \frac{2y}{1-y^2} = \tan^{-1} \frac{x+z}{1-xz}$$

$$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xz}$$

$$\Rightarrow \frac{x+z}{1-y^2} = \frac{x+z}{1-xz}$$

$$\Rightarrow (x+z) \left\{ \frac{1}{1-y^2} - \frac{1}{1-xz} \right\}$$

$$x+z=0 \text{ or } 1-xz=1-y^2$$

$$\Rightarrow y^2 = xz$$

i.e., x, y, z are in G.P.

Since x, y, z are in A.P. and G.P.

$$\therefore x = y = z$$

2. (1) $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Let $x = \tan \theta$

$$\therefore \frac{-\pi}{6} < \theta < \frac{\pi}{6}$$

$$\therefore \tan^{-1} y = \theta + \tan^{-1} \tan 2\theta = \theta + 2\theta = 3\theta$$

$$\therefore y = \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\therefore y = \frac{3x - x^3}{1 - 3x^2}$$

(1)

by (1)

So, $\alpha > \frac{\pi}{2}$ and $\beta > \pi$

$$\therefore \alpha + \beta > \frac{3\pi}{2}$$

2. (1), (2), (4)

$$f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{1}{1 + (x+j)(x+j-1)} \right)$$

$$= \sum_{j=1}^n \tan^{-1} \left(\frac{(x+j) - (x+j-1)}{1 + (x+j)(x+j-1)} \right)$$

$$= \sum_{j=1}^n [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$$

$$\therefore f_n(x) = \tan^{-1}(x+n) - \tan^{-1}(x)$$

(1) $f_n(0) = \tan^{-1}(n)$

$$\Rightarrow \tan^2(f_n(0)) = \tan^2(\tan^{-1} n) = n^2$$

$$\sum_{j=1}^5 \tan^2(f_j(0)) = \sum_{j=1}^5 j^2 = \frac{5 \times 6 \times 11}{6} = 55$$

(2) $f'_n(x) = \frac{1}{1 + (x+n)^2} - \frac{1}{1 + x^2}$

$$\Rightarrow f'_n(0) = \frac{1}{1+n^2} - 1$$

$$\Rightarrow 1 + f'_n(0) = \frac{1}{1+n^2}$$

Also, $\sec^2(f_n(0)) = \sec^2(\tan^{-1}(n)) = 1 + n^2$.

Hence, $(1 + f'_n(0)) \cdot \sec^2(f_n(0)) = \left(\frac{1}{1+n^2} \right) (1+n^2) = 1$

So, $\sum_{j=1}^{10} (1 + f'_j(0)) \sec^2(f_j(0)) = \sum_{i=1}^{10} 1 = 10$

(3) $\lim_{x \rightarrow \infty} \tan(f_n(x)) = \lim_{x \rightarrow \infty} \frac{n}{1 + x(n+x)} = 0$

(4) $\lim_{x \rightarrow \infty} \sec^2(f_n(x)) = \lim_{x \rightarrow \infty} (1 + \tan^2(f_n(x))) = 1$

Matrix Match Type

1. (b) - (p), (r)

$$\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$$

$$\Rightarrow \tan^{-1} \frac{(x+3) - (x-3)}{1 + (x^2-9)} = \tan^{-1} \frac{3}{4}$$

$$\Rightarrow \frac{6}{x^2-8} = \frac{3}{4}$$

$$\therefore x^2 - 8 = 8$$

$$\text{or } x = \pm 4$$

Clearly, both values satisfy the equation.

Note: Solutions of the remaining parts are given in their respective chapters.

Multiple Correct Answers Type

1. (2), (3), (4)

$$\alpha = 3 \sin^{-1} \frac{6}{11} > 3 \sin^{-1} \frac{6}{12} = 3 \sin^{-1} \frac{1}{2} = \frac{\pi}{2}$$

$$\text{and } \beta = 3 \cos^{-1} \frac{4}{9} > 3 \cos^{-1} \frac{4}{8} = \pi$$

$$\begin{aligned}
 (p) \quad & \frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \\
 &= \frac{\frac{1}{\sqrt{1+y^2}} + \frac{y^2}{\sqrt{1+y^2}}}{\frac{y}{\sqrt{1-y^2}} + \frac{y}{\sqrt{1-y^2}}} = \frac{\sqrt{1+y^2}}{y\sqrt{1-y^2}} \\
 &= y\sqrt{1-y^4}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \quad & \frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4 \\
 &= \frac{1}{y^2} (y^2(1-y^4)) + y^4 \\
 &= 1 - y^4 + y^4 = 1
 \end{aligned}$$

$$\begin{aligned}
 (q) \quad & \cot(\sin^{-1} \sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6})) \\
 \Rightarrow \quad & \cot\left(\cot^{-1} \frac{x}{\sqrt{1-x^2}}\right) = \sin\left(\sin^{-1} \frac{x\sqrt{6}}{\sqrt{1+6x^2}}\right) \\
 \Rightarrow \quad & \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{6x^2+1}} \\
 \Rightarrow \quad & 6x^2+1 = 6-6x^2 \\
 \Rightarrow \quad & 12x^2 = 5 \\
 \Rightarrow \quad & x = \sqrt{\frac{5}{12}} = \frac{1}{2}\sqrt{\frac{5}{3}}
 \end{aligned}$$

Note: Solutions of the remaining parts are given in their respective chapters.

$$\begin{aligned}
 3. (1) (s) \quad & \tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1} \frac{2}{x^2} \\
 \Rightarrow \quad & \tan^{-1}\left(\frac{3x+1}{4x^2+3x}\right) = \tan^{-1} \frac{2}{x^2} \\
 \Rightarrow \quad & 3x^2 - 7x - 6 = 0 \\
 \Rightarrow \quad & x = -\frac{2}{3}, 3
 \end{aligned}$$

But for $x = -\frac{2}{3}$, L.H.S. is negative and R.H.S. is positive.

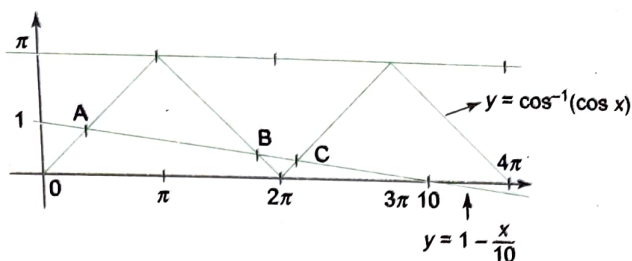
Hence, the only solution is $x = 3$.

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (3) $f: [0, 4\pi] \rightarrow [0, \pi], f(x) = \cos^{-1}(\cos x)$
 For number of roots of equation $\cos^{-1}(\cos x) = \frac{10-x}{10}$, draw the graph of $y = \cos^{-1}(\cos x)$

and $y = \frac{10-x}{10}$ and find the points of intersection.



From the graph number of solutions is 3.

2. (2) Given equation is

$$\begin{aligned}
 & \sin^{-1} \left(\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i \right) + \\
 & \cos^{-1} \left(\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i \right) = \frac{\pi}{2}
 \end{aligned}$$

$$\therefore \sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i$$

$$\Rightarrow \frac{x^2}{1-x} - x \frac{\frac{x}{2}}{1-\frac{x}{2}} = \frac{x}{1+x} + \frac{\left(-\frac{x}{2}\right)}{1+\frac{x}{2}}$$

(Each series is infinite G.P.)

$$\Rightarrow \frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{x}{1+x} - \frac{x}{2+x}$$

$$\Rightarrow x = 0 \text{ or } \frac{x}{1-x} - \frac{1}{1+x} = \frac{x}{2-x} - \frac{1}{2+x}$$

$$\Rightarrow x = 0 \text{ or } \frac{x+x^2-1+x}{1-x^2} = \frac{2x+x^2-2+x}{4-x^2}$$

$$\Rightarrow x = 0 \text{ or } (x^2+2x-1)(4-x^2) = (x^2+3x-2)(1-x^2)$$

$$\Rightarrow 4x^2+8x-4-x^4-2x^3+x^2 = x^2+3x-2-x^4-3x^3+2x^2$$

$$\Rightarrow x^3+2x^2+5x-2=0$$

$$\text{Let } f(x) = x^3+2x^2+5x-2$$

$$\therefore f'(x) = 3x^2+4x+5 > 0$$

So, $f(x)$ is increasing function and it intersects x -axis only once.

$$f(0) = -2 \text{ and } f(1/2) = 9/8.$$

So, one root of $f(x) = 0$ lies in $(0, 1/2)$.

So, there are two values of x .

2

APPENDIX

Chapterwise Solved January 2019 JEE Main Questions (All Sets)

Chapter 3

(Trigonometric Ratios and Transformation Formulas)

- For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, the expression $3(\sin \theta - \cos \theta)^4 + 6(\sin \theta + \cos \theta)^2 + 4 \sin^6 \theta$ equals
 - $13 - 4 \cos^6 \theta$
 - $13 - 4 \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta$
 - $13 - 4 \cos^2 \theta + 6 \cos^4 \theta$
 - $13 - 4 \cos^2 \theta + 6 \sin^2 \theta \cos^2 \theta$
- The maximum value of $3 \cos \theta + 5 \sin \left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is
 - $\sqrt{19}$
 - $\frac{\sqrt{79}}{2}$
 - $\sqrt{31}$
 - $\sqrt{34}$
- If $\sin^4 \alpha + 4 \cos^4 \beta + 2 = 4\sqrt{2} \sin \alpha \cos \beta$; $\alpha, \beta \in [0, \pi]$, then $\cos(\alpha + \beta) - \cos(\alpha - \beta)$ is equal to
 - 0
 - $-\sqrt{2}$
 - 1
 - $\sqrt{2}$

Chapter 4

(Trigonometric Equations)

- If $0 \leq x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$ is
 - 2
 - 1
 - 3
 - 4
- The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is
 - $\frac{\pi}{2}$
 - π
 - $\frac{3\pi}{8}$
 - $\frac{5\pi}{4}$

- In a triangle, the sum of lengths of two sides is x and the product of lengths of the same two sides is y . If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is
 - $\frac{y}{\sqrt{3}}$
 - $\frac{c}{\sqrt{3}}$
 - $\frac{c}{3}$
 - $\frac{3}{2}y$

- Given $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha} = \frac{\cos B}{\beta} = \frac{\cos C}{\gamma}$, then the ordered triad (α, β, γ) has a value
 - (3, 4, 5)
 - (19, 7, 25)
 - (7, 19, 25)
 - (5, 12, 13)

Chapter 6

(Height and Distance)

- Consider a triangular plot ABC with sides $AB = 7$ m, $BC = 5$ m and $CA = 6$ m. A vertical lamppost at the midpoint D of AC subtends an angle 30° at B . The height (in m) of the lamppost is
 - $7\sqrt{3}$
 - $\frac{2}{3}\sqrt{21}$
 - $\frac{3}{2}\sqrt{21}$
 - $2\sqrt{21}$
- If the angle of elevation of a cloud from a point P which is 25 m above a lake is 30° and the angle of depression of reflection of the cloud in the lake from P is 60° , then the height of the cloud (in metres) from the surface of the lake is
 - 42
 - 50
 - 45
 - 60

Chapter 7

(Inverse Trigonometric Functions)

13. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to

- (1) π (2) 7π (3) 0 (4) 10

14. The value of $\cot \left(\sum_{n=1}^{10} \cot^{-1} \left(1 + \sum_{p=1}^n 2p \right) \right)$ is

- (1) $\frac{22}{23}$ (2) $\frac{23}{22}$ (3) $\frac{21}{19}$ (4) $\frac{19}{21}$

15. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval

- (1) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$
(2) $(\cot 5, \cot 4)$

(3) $(\cot 2, \infty)$

(4) $(-\infty, \cot 5) \cup (\cot 2, \cot \infty)$

16. Considering only the principal values of inverse functions, the set $A = \left\{ x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4} \right\}$

- (1) is an empty set
(2) contains more than two elements
(3) contains two elements
(4) is a singleton set

Answers Key

- | | | | | | | | | | |
|--------|--------|--------|--------|---------|---------|---------|---------|---------|---------|
| 1. (1) | 2. (1) | 3. (2) | 4. (1) | 5. (1) | 11. (2) | 12. (1) | 13. (1) | 14. (3) | 15. (4) |
| 6. (2) | 7. (1) | 8. (2) | 9. (3) | 10. (2) | 16. (4) | | | | |